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ELEMENTARY ALGEBRA

NEUFELD

LEONHARD EULER
(From an Engraving of the Painting by Darbes)

Born at Basel, 1707. Died at St. Petersburg, 1783

Euler was physicist, astronomer and mathematician.

He contributed greatly to the progress of higher mathematics, but, in his old age, after he had become blind, he dictated his *Anleitung zur Algebra*, 1770, which is purely elementary and one of the earliest works systematizing the fundamental processes of Algebra.

ELEMENTARY ALGEBRA

WITH A

TABLE OF LOGARITHMS

BY

J. L. NEUFELD
CENTRAL HIGH SCHOOL OF PHILADELPHIA, PA.

SIX PORTRAITS
AND
TWENTY-EIGHT ILLUSTRATIONS

PHILADELPHIA
P. BLAKISTON'S SON & CO.
1012 WALNUT STREET

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PREFACE

This *Elementary Algebra* is designed to meet the demand for a text book that is both elementary and comprehensive. Elementary, in that it is written to conform to the order of intelligence usually found in beginners; comprehensive, in that the ground covered is sufficient to meet college entrance requirements.

In many text books, authors have sacrificed clearness for conciseness, with the result that students are unable to prepare their lessons, and teachers are obliged to lose much time and energy in doing the work the text should have done for them. This is particularly noticeable in reviewing.

This text contains an abundance of explanation, identical with that used by the author, in his class room, for more than a quarter of a century and it should be sufficiently clear, even to the slowest student.

The similarity of Algebra to Arithmetic has been emphasized throughout the book, in order to bridge the gap between these subjects. The student is made to see that the rules of Arithmetic are also used in Algebra.

The various topics are arranged in an order which the author's experience has taught him to be best for High School students. The exercises are carefully graded and selected, and especially difficult problems have been avoided. The demand for practical problems has been met as far as possible, but problems requiring an exposition of various principles of the sciences, for which the teacher may have neither time nor inclination, have been excluded. The Review Exercises are made up of questions taken from High School examination papers.

The author does not agree with so-called modern educators who think that too much space is given to the subject of Factoring in text books. A thorough knowledge of Factoring gives the student a most desirable training in inspection methods, the methods of highest practical value to students of Algebra. Most fail~~ures in Algebra can~~ be directly traced to ignorance of Factoring, ~~and~~ the author has therefore given this topic the ~~space its importance de~~mands.

Memory aids are introduced, throughout the book, to help students in memorizing long rules, such as the rule for extracting cube roots and the various rules in the theory of exponents.

No startling innovations will be found, but a number of topics, as for instance, the subject of Graphs, have been treated in a novel way.

The various portraits of eminent mathematicians, with biographical and historical notes, have been introduced to add a touch of human interest and, perhaps, lead students to a further investigation into the history of mathematics.

The author believes that teachers of mathematics will find the book eminently adapted to their work, and hopes that it will prove to be a real teacher's helper. The student's excuse, "I couldn't understand the explanation in the book," should disappear.

J. L. NEUFELD.

CENTRAL HIGH SCHOOL, PHILADELPHIA.

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CHAPTER I

INTRODUCTION AND EVALUATION

1. Algebra is merely an extension or continuation of Arithmetic. The rules, principles and methods of Arithmetic are also used in Algebra. The one great difference between them is, that while in Arithmetic, numbers are expressed by the figures or digits, 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9, in Algebra, we use figures and letters to represent these ~~numbers. By~~ employing these figures and letters, we are able to solve problems with less work ~~and in~~ many instances, solve problems that could not be solved ~~by arithmetic~~ alone. Any letters of the alphabet may be used, but it is usual to represent **known numbers** by those in the beginning of the alphabet as *a*, *b*, *c*, and to represent **unknown numbers,** that is, numbers whose values are to be found, by the letters at the end of the alphabet, as, x, y, z.

Quantity.—A number of certain units is called a quantity. Thus 5 *cents*, 6 *pounds*, 3 *feet* are called quantities. In arithmetic, these quantities were called **concrete numbers,** in distinction from numbers without names, as 5, 12, or 7, which were called **abstract numbers.**

2. Signs of Operation.—The principal signs of operation are the same in Algebra ~~as~~ in Arithmetic. The **Sign of Addition,** +, which is read *plus*, meaning more, shows that the numbers connected ~~by this sign~~ are to be added. Thus, 3 + 4, shows that the numbers 3 and 4 are to be added. In the same way, $a + b$, means that the numbers which a and b represent are to be added. In the case of ~~the~~ numbers, 3 and 4, this addition can actually be performed by combining them, giving us the sum 7, but when we are asked to add the numbers a and b, we cannot combine them, but only indicate the addition and write as our sum, $a + b$.

The **Sign of Subtraction,** $-$, read *minus*, shows that the second number is to be subtracted from the first. Thus, $7 - 4$, which is read, seven minus four, means that 4 is to be subtracted from 7, and the difference is 3. In the same way, $a - b$, read *a minus b*, means that b is to be subtracted from a. Here, as in the similar case in addition, the numbers a and b cannot be combined and we indicate the subtraction, calling the difference $a - b$.

The **Sign of Multiplication,** $\times$, read *multiplied by* or *times*, shows that the first number is to be multiplied by the second. Thus, 4×3, is read *four times three*, or *four multiplied by three*, and $a \times b$, is read *a multiplied by b* or *a times b*. We can actually multiply 4 by 3 and obtain the product 12, but in multiplying a by b, we have the same difficulty as before, and can only indicate the product, $a \times b$. Where letters are used to represent numbers, that is, in the case of **literal numbers,** we frequently use a dot, $\cdot$, placed between the numbers and above the line, to distinguish it from a decimal point, to indicate the multiplication as, $a \cdot b$, which means $a \times b$, or the sign may be omitted altogether and the letters written beside each other, as ab, which also means, $a \times b$.

The **Sign of Division,** $\div$, read *divided by*, shows that the first number is to be divided by the second number. Thus, $8 \div 2$, read *eight divided by two*, means that 8 is to be divided by 2, and, $a \div b$, read, *a divided by b*, means that the number a is to be divided by the number b. The division of a by b can only be indicated and the quotient is written $a \div b$. As every fraction is an indicated division, the numerator being the dividend and the denominator, the divisor, $a \div b$ may also be written as $\frac{a}{b}$ or a/b.

OTHER SIGNS USED IN ALGEBRA

3. The **Sign of Equality,** $=$, is read *is equal to* or *equals* and means that the two numbers, between which it is placed, are equal. Thus $4 + 7 = 11$, is read, four plus seven is equal to eleven and means that the sum of the numbers 4 and 7 is equal to the number 11. Also, $a + b = 6$, is read *a plus b equals* 6 and means that the sum of the numbers a and b is equal to the number 6; $x = a + b$, means that the number x is equal to the number, $a + b$.

4. The **Signs of Aggregation.**—Since, in the case of literal numbers, the arithmetical operations of addition, subtraction, multiplication and division cannot be performed by actually combining them, it is frequently found necessary to treat such indicated operations as single numbers. Thus, the sum, $a + b + c$, is to be multiplied by 3. To show that $a + b + c$ is to be treated as a single number, we place a **parenthesis** around it, thus $(a + b + c)$ and to indicate the multiplication by 3, we simply connect the two numbers, 3 and $(a + b + c)$ by the usual sign of multiplication, $3 \times (a + b + c)$ or $3(a + b + c)$, no sign between the numbers, indicating a multiplication. In addition to the parenthesis, ( ), we have the **Bracket** [ ], the **Brace** { }, the **Vinculum** —— and the **Bar** |. All these signs perform the same duty of *aggregating*, that is, taking any number of figures and letters together, to be treated as a single number. Thus $x \div [b - c]$, means that the number x is to be divided by the number $[b - c]$; $\{ab + x\} \div y$, means that the number $\{ab + x\}$ is to be divided by the number y; $c - \overline{x + y}$, means that the number $\overline{x + y}$ is to be subtracted from the number c. The value of these signs can readily be appreciated. Suppose we omitted the vinculum in the last expression and wrote it, $c - x + y$; this would mean that the number x is to be subtracted from the number c and the number y added to the resulting number.

5. Factors.—The parts of a number, which, when multiplied, produce the number, are called its **Factors.** Thus, 3 and 4 are factors of 12, because 3×4 will produce 12; $6xy$ will have as its factors, 3, 2, x and y, because $3 \times 2 \times x \times y$ equals the number $6xy$; similarly, the factors of $c(a + b)$ are c and $(a + b)$, since their product will produce

$c(a+b)$. Factors, expressed by figures, are called **numerical factors**, and those, expressed by letters, are called **literal factors**.

6. Coefficients.—Any factor of a number may be considered a multiplier of the other factors and is then called the **coefficient** or co-factor of this number. Thus, in the number $6ab$, 6 may be called the coefficient of ab, or, a may be called the coefficient of $6b$, or b may be called the coefficient of $6a$. Here again we distinguish between **numerical coefficients** and **literal coefficients.** When a literal number has no figure preceding it, the numerical coefficient 1 is understood. Thus, xy has the numerical coefficient 1 understood, since xy is equal to $1 \times xy$.

7. Powers.—When a number is the product of two or more equal factors, it is called **a power** of the factor. Since 8 is equal to $2 \times 2 \times 2$, it is called a power of 2. Similarly, $a \times a$ is called a power of a. To avoid writing all the equal factors of such a product, a small figure, called the **exponent,** is written to the right and slightly above one of these factors, showing how many equal factors the product contains. Thus, 2^3 means $2 \times 2 \times 2$ and a^2 means $a \times a$. These powers are named according to the number of such equal factors. Thus a^2 is called the **second power** of a; 2^3 is the **third power** of 2, b^4 is the **fourth power** of b, x^{10} is the **tenth power** of x.

The second power of any factor, as b^2 is called the **square** of b, because if b represents the number of units of length in the side of a square, $b \times b$ will give the number of square units or units of area in the square. For a similar reason, x^3 is called the **cube** of x, since, if x represents the number of units of length in the side of a cube, $x \cdot x \cdot x$ represents the number of units of volume in that cube.

Care should be taken to distinguish the meanings of coefficient and exponent. Thus, in $3x$, 3 is the coefficient of x and shows that three x's are added to produce $3x$, that is $3x = x + x + x$, but the exponent 3 of x^3 shows that three x's are multiplied together, that is $x^3 = x \cdot x \cdot x$.

8. Roots.—When a number is the product of two or more equal factors, each of these factors is called a **root** of that number. Thus 2 is a root of 8, x is a root of x^2. A root of any number is indicated by the **root sign** or **radical sign,** $\sqrt{\ }$, placed over that number, and a small figure, placed in the sign, called the **index** of the root, shows what root of the number is to be taken. Thus $\sqrt[3]{x^6}$ means the **third** or **cube root** of x^6; $\sqrt[5]{y^5}$, means the **fifth root** of y^5. If no index is written, 2 is understood, thus the square root of a is written, $\sqrt{a}$, the index 2 being understood.

9. Algebraic Expressions.—Any number written with algebraic symbols is called an **algebraic expression** and may consist of one symbol or two or more symbols connected by signs of operation. Thus, x, $2ab$, $r + st$, are algebraic expressions.

Terms.—The parts of an algebraic expression, which are separated from each other by either a plus sign or a minus sign, are called the **terms** of that expression. Thus, in the expression $2ab + \frac{b^2}{c}$, $2ab$ and $\frac{b^2}{c}$ are the two terms. Note that the *parts of a term* may be connected by signs of multiplication or division, but not by the signs + or −.

Similar Terms.—The terms of an algebraic expression, which contain the same letters with the same exponents, are called **similar terms** or **like terms.** Thus, $2a^2b^3$, a^2b^3, $7a^2b^3$ are similar terms.

Monomial.—An algebraic expression consisting of only one term, is called **a monomial** or **simple expression.** Thus, xy, a^2b, or $\frac{2s}{t}$ are monomials or simple expressions.

Polynomials.—An algebraic expression of two or more terms is called a **polynomial** or **compound expression.** Thus, $ut + \frac{1}{2}at^2$, $a^2 + 2ab + b^2$, are polynomials. An expression of two terms is called a **binomial** and one of three terms, a **trinomial.**

10. Numerical Value of an Expression.—If in an algebraic expression, particular values are placed for the letters and the indicated operations are performed, we obtain the **numerical value** of the expression.

1. If $x = 2$, find the numerical value of $6x$ and $3x^3$.
Substituting the value 2 for x,

$$6x = 6 \times 2 = 12.$$
$$3x^3 = 3 \times 2 \times 2 \times 2 = 24.$$

2. If $a = 4$, $t = 5$, find the numerical value of $\frac{1}{2}at^2$.
Substituting the values of a and t,

$$\tfrac{1}{2}at^2 = \tfrac{1}{2} \times 4 \times 5 \times 5 = 50.$$

3. If $a = 2$, $b = 3$, $c = 4$, find the numerical value of $\sqrt{\frac{4a^4b^2}{c}}$.
Substituting the values of a, b and c,

$$\sqrt{\frac{4a^4b^2}{c}} = \sqrt{\frac{\cancel{4} \times 2 \times 2 \times 2 \times 2 \times 3 \times 3}{\cancel{4}}} = \sqrt{144} = 12.$$

4. If $u = 2$, $v = 5$, find the numerical value of $\sqrt{9}uv$. As no vinculum or parenthesis or other sign of aggregation is used here, the radical sign belongs only to the symbol immediately following it. Note the difference between this problem and the preceding.

$$\sqrt{9}uv = \sqrt{9} \times 2 \times 5 = 3 \times 2 \times 5 = 30.$$

Note.—If one factor of a product is equal to zero, the product will equal zero, since any number times zero is equal to zero.

EXERCISE 1

If $a = 1, b = 2, c = 3, d = 4, x = 5, y = 6, z = 0$, find the numerical values of:—

1. $7x$.
2. $3ab$.
3. $4c^2$.
4. $2x^2y$.
5. $6a^3b^2$.
6. ab^2cx.
7. $\frac{1}{2}ab$.
8. $\frac{3}{4}a^2x$.
9. $\frac{7}{8}b^2cy$.
10. $\frac{3}{5}a^2b^2z^3$.
11. $\frac{5xy}{3ab}$.
12. $\frac{2a^2b^2}{3xy^3}$.
13. $\sqrt{9}ab$.
14. $\sqrt{4ab}$.
15. $\sqrt[3]{}27x^2y$.
16. $\sqrt[3]{bdx^3}$.
17. $\sqrt{4cdxz}$.
18. $\sqrt{}(3cx^2)$.
19. $3\sqrt{2bc^2}$.
20. $\frac{5}{2}\sqrt[3]{bd}$.
21. $x\sqrt{abcy}$.
22. $\frac{1}{2}y\sqrt[3]{ac^3}$.

NUMERICAL VALUES OF POLYNOMIALS

11. In finding the numerical value of a polynomial, find the value of each term, as in the preceding exercise, and then add or subtract the several terms as indicated by the signs + or − between the terms. In algebra, it is usual to proceed from left to right, not only in finding the value of each term, but also in combining the several terms.

It is evident, that the terms of an algebraic expression may be arranged in any order without altering the value of the expression. This is called the **commutative law** for addition and subtraction.

Thus, $4 + 6 + 3$ is equal to $6 + 4 + 3$ or to $3 + 6 + 4$.

1. If $a = 2$, $b = 3$, $c = 4$, find the value of $a^2 + 2bc - ac$.
Substituting the values of the letters,

$$a^2 + 2bc - ac = 2^2 + 2 \times 3 \times 4 - 2 \times 4 = 4 + 24 - 8 = 20.$$

2. If $x = 5$, $y = 4$, $z = 0$, find the value of $(x + z)(x + y) + \sqrt{y}$.
$(x + z)(x + y) + \sqrt{y} = (5 + 0)(5 + 4) + \sqrt{4} = 5 \times 9 + 2 = 45 + 2 = 47$.

3. If $a = 1$, $b = 2$, $c = 3$, $d = 5$, $x = 0$, find value of

$$\frac{a+b}{b+c}\sqrt{d^2} - \tfrac{1}{2}bc^2 \div (c - 2a) - 5bcdx$$

substituting the values of the letters,

$$\frac{1+2}{2+3} \times \sqrt{5^2} - \tfrac{1}{2} \times 2 \times 3^2 \div (5 - 2 \times 1) - 5 \times 2 \times 3 \times 5 \times 0 =$$
$$\tfrac{3}{5} \times \sqrt{25} - \tfrac{1}{2} \times 2 \times 9 \div (5 - 2) - 0 = \tfrac{3}{5} \times 5 - 9 \div 3 =$$
$$3 - 3 = 0.$$

EXERCISE 2

If $a = 2$, $b = 3$, $c = 4$, $d = 5$, $x = 6$, $y = 3$, $z = 0$, find the value of:

1. $2x + 3y - 4z$.
2. $a^2 - 2ab + b^2$.
3. $\frac{1}{2}xy\sqrt{c} - x\sqrt[3]{ac}$.
4. $(a + b)(a + c) - 2x \div y$.
5. $\sqrt{(a + c)x}$.
6. $(a + b)^2 - (x - c)^2$.
7. $xy \div (d - b) - (x - a)(y + z)$.
8. $\frac{1}{7}(c^3 - a^3) - a - x$.
9. $\sqrt{4cd} \div (c + x) + xy - (b - a)$.
10. $\sqrt{bcy} \div (d - a) + \sqrt[3]{8ac}$.

REVIEW EXERCISE I

1. When $a = 5$, $b = 3$, $c = 10$, $m = 4$, $n = 1$, $k = 0$, find value of

$$c + \frac{2m}{c - 2m} + \sqrt{a^2b^2} - abk.$$

2. If $m = 2$, $s = 3$, $p = 1$, $z = 0$, find value of

$$(m + s)^2 + 2msp + \frac{m^2}{p - s}.$$

3. If $x = 6$, $y = 3$, $z = 0$, $r = 2$, $s = 10$, find the value of

$$(x - y)^2 + 2(x - y)(x + y) + (r + s)^2 + \frac{7\,xyz^2}{4rs}.$$

4. If $a = 1$, $b = 2$, $c = 3$, $d = 4$, find value of

$$\frac{a^2(d - c)}{\sqrt{c^2 + d^2}} - \frac{3(d + b)^2}{c - b}.$$

5. If $a = 3$, $b = 2$, $x = 12$, find value of

$$\frac{ab + 2x - 10}{4a + 4b} - \frac{3abx - ab}{ab + bx}.$$

6. If $a = 1$, $m = 13$, $n = 14$, $x = 25$, $y = 24$, find the value of

$$\frac{9 + 2y + 3m}{x - a} - \sqrt{\frac{2m + \frac{1}{3}ny + 6}{m^2}}.$$

7. If $a = 5$, $b = 2$, $c = 3$, $d = 4$, find the value of

$$(a - b)^2 + (2a - 3b)^2 - (b + c)^2.$$

8. When $a = 4$, $b = 5$, $m = 6$, $n = 10$, find the value of

$$\frac{n}{\sqrt{n - m}} - \frac{6m}{\sqrt{4a}} + \sqrt{3n - b}.$$

9. If $a = 1$, $b = 2$, $c = 3$, $d = 4$, $e = 5$, $f = 6$, find the value of

$$be + d\sqrt{cd + d} - (f - d)\sqrt[3]{cd - d}.$$

10. If $a = 1$, $b = 2$, $c = 3$, $d = 0$, find the value of

$$2b + \frac{5c - 3}{2c - a} + 9\sqrt{abd^2}.$$

11. When $a = 5$, $b = 2$, $n = 1$, $x = 4$, find the value of

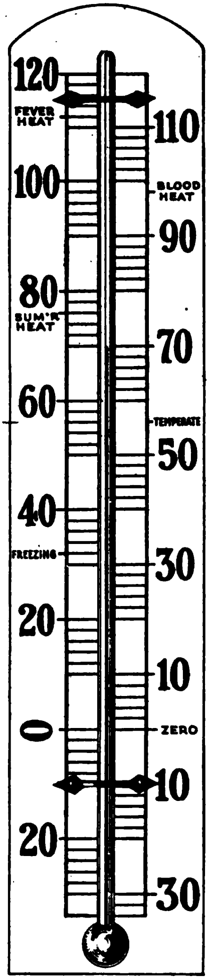
$$\frac{a^3 - 10b^3}{3a} \text{ and } \frac{75 - ab^2n^3}{x^2 - a}.$$

12. When $a = 5$, $b = 3$, $c = 10$, $m = 4$, find the value of

$$m^{a-b} \quad \text{and} \quad \sqrt[3]{4ac^2m}.$$

CHAPTER II

POSITIVE AND NEGATIVE NUMBERS, PARENTHESIS



12. We frequently find quantities that are opposed to each other. For instance, a rise in temperature is of opposite kind to a fall in temperature; a gain in business is opposed to a loss; travelling a certain distance north is opposed to a distance going south. These quantities, which oppose each other and which may be called of opposite kind, are not considered in Arithmetic, but must be cared for in Algebra. We distinguish these by calling one quantity **plus** or **positive** and its opposite, **minus** or **negative.** Thus, on a thermometer, degrees of temperature above the zero point are called *plus* or *positive* and degrees below zero are called *minus* or *negative;* + 10° means 10 degrees above zero and − 8° means 8 degrees below zero. Either of two opposite quantities may be called the positive and the other the negative; but it is usual to call that quantity positive which implies an increase or gain. For instance, if we were climbing a steep mountain side and occasionally slipping back a certain distance, those distances that bring us

RENÉ DESCARTES DU PERRON

(From an Old Engraving of the Painting by Frans Hals)

Born at La Haye, in Touraine, 1596. Died at Stockholm, 1650. René Descartes was philosopher, physicist and mathematician. The interpretation and systematic use of negative quantities began with Descartes.

He discarded Vieta's algebraic notation and devised the symbolism now used.

He is also the discoverer of analytic geometry.

nearer the top of the mountain would be called plus or positive and the distances we slip backwards would be minus or negative.

Only one kind of quantities are considered in Arithmetic and they are measured by stating the number of units they contain. For this purpose, we use the *Natural Series* of *Numbers* 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, and so on, numbers starting with zero and increasing by one as far as we please. This series of numbers is no longer sufficient, when we consider positive and negative quantities. Suppose the thermometer indicates 10° and there is a drop in temperature amounting to 15°. Counting backwards from 10, in the natural series of numbers, we reach zero but still have 5 degrees drop to count. To enable us to count this negative temperature, we extend the natural series of numbers, beginning at zero and counting negative units in the opposite direction, that is, to the left. We then have the *Algebraic Series of Numbers.*

$$\ldots -10,\ -9,\ -8,\ -7,\ -6,\ -5,\ -4,\ -3,\ -2,\ -1,\ 0,\ +1,\ +2,\ +3,\ +4,\ +5,\ +6,\ +7,\ +8, \ldots$$

Using this algebraic series of numbers, it is evident that a drop of 15° from 10° brings us to $-5°$ or 5 degrees below zero.

Similarly, if a man has a capital of $5000 and gains $1000, he will have $6000, but if he then loses $8000, his capital would become equal to $2000 less than zero, or −$2000, which means that he is $2000 in debt.

We have here one instance of the extension of arithmetical methods, as the algebraic series of numbers enables us to subtract a larger from a smaller number, which we were unable to do in Arithmetic.

13. The **Absolute Value** of a number is its value without regard to its sign. Thus $+5$ and -5 have the same absolute value of 5.

It should be noted that, when no sign is written before a number, plus is understood. The minus sign, indicating a negative number, must always be written.

14. Double Meanings of the + and − Signs.—From the foregoing explanations, it should now be clear that the plus and minus signs not only indicate an addition or subtraction as in Arithmetic, but also show that the numbers or quantities are of opposite kinds. These two meanings of the signs do not contradict each other. If we write $8 - 5$, indicating that 5 should be subtracted from 8, the result of the subtraction, 3, is the same, as if we had combined 8 positive units with 5 negative units, that is $8 + (-5)$. In the subtraction, the 5 units destroy 5 of the 8 units and leave 3 units. Similarly, since -5 is of opposite kind to $+8$, five of the $+8$ units are cancelled or balanced by the five negative units and we have 3 positive units left. Two numbers of opposite kind, that is, one plus and the other minus, are said to have *unlike signs*. Thus, xy and $-ab$, have unlike signs.

PARENTHESIS

15. Parenthesis Preceded by + Sign.—If a man, in an automobile, travelled 100 miles the first day, and on the second day, went 40 miles in the morning and 30 miles in the afternoon, the total distance travelled in the two days can be found, either by successively adding to the first day's distance, the distance he went in the morning and then the distance he went in the afternoon or by adding to the first day's distance, the distance he travelled the second day. The two methods may be written,

$$100 + 40 + 30 \qquad \text{(first method)}$$

$$\text{or} \quad 100 + (40 + 30) \qquad \text{(second method)}$$

showing that, $100 + (40 + 30) = 100 + 40 + 30.$
Similarly, $a + (b + c) = a + b + c.$

If a man in an automobile travels 100 miles the first day, 40 miles in the morning of the second day but travels back 30 miles in the afternoon, then, at the end of the second day, the distance from his starting point can be found, either by adding to the first day's distance, the distance he travelled in the morning of the second day and then subtracting the distance he returned in the afternoon or by adding to the first day's distance, the difference between the distances travelled in the morning and afternoon of the second day.

By the first method, we get, $100 + 40 - 30 = 110$

By the second method, we get, $100 + (40 - 30) = 110$

showing that, $100 + (40 - 30) = 100 + 40 - 30.$

and similarly, $a + (b - c) = a + b - c.$

We have, from the above, the general

Rule for Removal of Parenthesis Preceded by + Sign.—*A Parenthesis, preceded by the sign +, may be removed without making any change in the signs of the terms of the expression.*

16. Parenthesis Preceded by − Sign.—If a man, starting at a certain place, travels 100 miles the first day but returns, 40 miles in the morning of the second day and 30 miles in the afternoon, his distance from the starting point at the end of the second day, may be found either by successively subtracting from the first day's distance, the distances he returned in the morning and afternoon of the second day, or by subtracting from the first day's distance, the sum of the distances he returned on the second day. This may be written,

First method $100 - 40 - 30 = 30.$

Second method $100 - (40 + 30) = 30.$

Therefore $100 - (40 + 30) = 100 - 40 - 30.$

Similarly, $a - (b + c) = a - b - c.$

If a man, starting at a certain place, travels 100 miles the first day, returns 40 miles in the morning of the second day and then, in the afternoon, finds it necessary again to travel 30 miles away from his starting point, the distance from his starting point at the end of the second day, may be found either by subtracting from the distance he travelled the first day, the distance he returned on the morning of the second day and then adding the distance he went in the afternoon, or by subtracting from the first day's distance, the difference between the two distances he travelled in the morning and afternoon of the second day.

This may be written,

By the first method $100 - 40 + 30 = 90$.

By the second method, $100 - (40 - 30) = 90$.

Showing that, $100 - (40 - 30) = 100 - 40 + 30$.

Similarly, $a - (b - c) = a - b + c$.

We have, therefore, the general rule for

Removal of a Parenthesis Preceded by the Sign −.

A Parenthesis, preceded by a − sign, may be removed by changing the signs of the terms that were in the parenthesis.

Note.—The two rules, for the removal of a parenthesis, apply equally to the removal of any of the signs of aggregation, as the bracket, brace or vinculum. In the case of the vinculum, the sign before the first of the terms under the vinculum is understood to be the sign before the vinculum.

Thus, $a - \overline{b + c}$ is equivalent to $a - (b + c)$.

EXERCISE 3

Remove the parenthesis and combine like terms:

1. $7 + (4 + 2)$.
2. $7 + (4 - 2)$.
3. $5 - (3 - 2)$.
4. $8 - (4 + 2)$.
5. $(4 + 2) + (3 + 4)$.
6. $(4 - 2) + (6 - 3)$.
7. $(5 + 3) - (2 + 4)$.
8. $(3 + 2) - (5 - 6)$.
9. $(7 - 3) - (7 - 9)$.
10. $9 - (8 + 3 - 5)$.

If $a = 6$, $b = 5$, $c = 4$, $d = 3$, find the value of:

11. $a + (b + c)$.
12. $b + (a - c)$.
13. $(a + b) - (c + d)$.
14. $(a - b) - (c - d)$.

PRODUCT OF A COMPOUND BY A SIMPLE FACTOR

17. In finding the product of $3(4 + 2)$, we get the same result, whether we multiply the sum of 4 and 2 by 3, or multiply 4 by 3 and 2 by 3 and then add the products.

By the first method, $3(4 + 2) = 3 \times 6 = 18$

By the second method, $3(4 + 2) = 3 \times 4 + 3 \times 2 = 12 + 6 = 18$

In the same way,

$$3(5 - 2) = 3 \times 3 = 9$$
$$\text{or } 3(5 - 2) = 3 \times 5 - 3 \times 2 = 15 - 6 = 9$$

If we use algebraic symbols, as $a(b + c)$, the second method must be used, since we cannot combine b and c by addition, as in the case of numbers like 5 and 2.

Thus $a(b + c) = ab + ac$

And $a(b - c) = ab - ac$

We call this the **distributive law** for **multiplication.**

In finding the product of any number of factors, these factors may be used in any order, without affecting the result.

Thus $3 \times 4 \times 5 = 4 \times 3 \times 5 = 5 \times 3 \times 4 = 60$

Similarly $2(4 - 2) = (4 - 2)\ 2 = 4$

And $abc = acb = bca$

We call this the **commutative law** for **multiplication.**

Perform the indicated operations:

1. $r + u(a + v)$
   $r + u(a + v) = r + (ua + uv) = r + ua + uv.$
2. $s + 2(u - v)$
   $s + 2(u - v) = s + (2u - 2v) = s + 2u - 2v.$
3. $t - 4(r + s)$
   $t - 4(r + s) = t - (4r + 4s) = t - 4r - 4s.$
4. $(x + y)z - v(u + t)$
   $(x + y)z - v(u + t) = (xz + yz) - (uv + ut) = xz + yz - uv - ut.$

EXERCISE 4

Perform the indicated operations and find the numerical value of each expression, when $a = 2, b = 3, c = 4, x = 5, y = 6, z = 0$.

1. $c + a(x + y)$.
2. $a + b(y - x)$.
3. $b - a(y + z)$.
4. $a - b(x - y)$.
5. $(b - a)c - (x - y)$.
6. $(x + a)b + c(y - z)$.
7. $a(x + y) - b(x - y)$.
8. $a^2b - c(x - y)$.
9. $bc - a^2(x - c)$.
10. $4(x^2 - a^2) + bc$.
11. $(c^2 - a^2)b - a(c^2 - b^2)$.
12. $(x^2 - z^2)c + b(y^2 - a^2)$.

QUOTIENT OF A COMPOUND BY A SIMPLE EXPRESSION

18. In finding the quotient of $(9 + 6) \div 3$, we obtain the same result, whether we divide the sum of 9 and 6 by 3 or divide 9 by 3 and 6 by 3 and then add the quotients.

By the first method $(9 + 6) \div 3 = 15 \div 3 = 5$

By the second method $(9 + 6) \div 3 = 9 \div 3 + 6 \div 3 = 3 + 2 = 5$

Similarly, $(9 - 6) \div 3 = 3 \div 3 = 1$

or $(9 - 6) \div 3 = 9 \div 3 - 6 \div 3 = 3 - 2 = 1$

and also, $(a - b) \div c = a \div c - b \div c = \frac{a}{c} - \frac{b}{c}$

This is called the **distributive law for division.**

Perform the indicated operations:

1. $a + (4a + 2b) \div 2$

$$a + (4a + 2b) \div 2 = a + \left(\frac{4a}{2} + \frac{2b}{2}\right) = a + (2a + b) = a + 2a + b = 3a + b.$$

2. $(ac + bc) \div c$

$$(ac + bc) \div c = \frac{ac}{c} + \frac{bc}{c} = a + b.$$

3. $(a^2 - c^2) \div b^2 = \frac{a^2}{b^2} - \frac{c^2}{b^2}.$

Note that the value of each term should be found first and then like terms combined.

EXERCISE 5

Perform the indicated operations and find the numerical value of each expression, when $a = 9$, $b = 6$, $c = 3$.

1. $(a + b) \div c.$
2. $(a - c) \div b.$
3. $(ac + bc) \div b.$
4. $(ac - bc) \div c.$
5. $(bc + b^2) \div a.$
6. $(a^2 - b^2) \div c^2.$
7. $(b^2 - bc) \div bc.$
8. $(a - b) \div c + c(a + b).$
9. $a^2 - (b + c) \div c.$
10. $(a^2 - b^2) \div c^2 - (b^2 + c^2) \div a.$

REVIEW EXERCISE II

1. If $x = 2$, $y = -3$, $z = 1$, $w = 0$, find the value of

$$\frac{4xy - 5z + 2x^2z - 5x^3w - 7w^2}{y^3}.$$

2. When $a = 1$, $b = 2$, $c = 3$, $d = 4$, $e = 5$, find the value of

$$a - (e + b) - (c + d) - (e - d + b + c).$$

3. If $a = 12$, $b = 10$, $x = 5$, $y = 8$, $z = -2$, find the value of

$$(a - b)^2 + (x - y)^2 - (a + z)^2.$$

4. If $a = 5$, $b = -2$, $n = 0$, $x = 3$, find the value of

$$ax^2 - \frac{5abx}{6} + 7bm + 15b - 7.$$

5. If $a = 3$, $b = 2$, $c = -1$, $d = 0$, find the value of

$$\frac{a}{\sqrt{a^2 + 8b}} + 2abc - \frac{a^2d}{bc}.$$

6. Find the value of $x^3 + y^3 + z^3 - 3xyz$, if $x = 1$, $y = 2$, $z = -3$.

7. If $x = -2$, $y = 4$, $a = 0$, $d = -1$, $m = 3$, find the value of

$$my^2 - (y - x) + axy^2 + (x - y)^2 + d^3.$$

8. If $x = -3$, $y = 2$, $z = 1$, $c = 0$, find the value of

$$xyz^2 - (x - y)(y - z) - y^{3z} + (y + c) - z^y + \frac{2y + x}{z}.$$

9. If $a = 1$, $b = -1$, $c = 2$, $d = 0$, find the value of

$$7a^2bc - (2a - b) + (a - c)^2 + 14abcd + a^{c+1}.$$

10. If $a = -3$, $b = 2$, $c = 1$, $d = 0$, $x = 4$, find value of

$$x^b - \frac{ac^2}{b + c} - (a + b)^2 + 3b^2c - \tfrac{1}{2}\,bcd.$$

CHAPTER III

SIMPLE EQUATIONS

19. Equation.—An equation is a statement of equality between two algebraic expressions.

Thus, if we wish to state that the expression $2x + 3$ is equal in value to the expression $3x + 2$, we write this statement as an equation, $3x + 2 = 2x + 3$. This means that the two numbers, represented by the two expressions, $3x + 2$ and $2x + 3$, are equal numbers or the same number.

The two equal expressions are called the **members** or **sides** of the equation. The one, to the left of the sign of equality, is the **left member** and the other, the **right member** of the equation.

If the statement of equality is true for all values of the letters used, the equation is called an **identical equation** or simply an **identity.**

Thus, the equation, $a + b = b + a$ is an identity since the two members of the equation will have equal numerical values for any values of a and b. It is also evident, that any equation that does not contain letters will be an identity, as, $5 + 3 = 4 \times 2$.

If we consider the equation, $3x + 2 = 2x + 3$, the statement may be read, *three times the number x plus two must equal twice the number x plus three.* In other words, the value of x must be such as to agree with the condition expressed by this equation. It can easily be seen that the two members will only be equal if x is equal to 1.

Such an equation is therefore called an **equation of condition** or simply an **equation.**

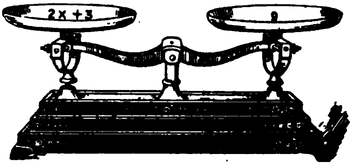
Since this value of x satisfies the condition expressed by the equation, we say that the equation is **satisfied** by this value of x.

Equations are usually employed in finding the value of some unknown number from its relation to certain known numbers. This relation is expressed algebraically by the equation. Unknown numbers are usually represented by the letters at the end of the alphabet, as x, y, z, and the known numbers either by the Arabic numerals or the letters at the beginning of the alphabet, as a, b, c.

20. Simple Equation.—When an equation is reduced to its simplest form and then contains only the **first power** of the unknown numbers, it is called a **simple equation** or **equation of the first degree.** Thus, $2x + 3 = 9$ and $ax - b = c$ are simple equations in x.

Solution of Equations.—Finding that value of the unknown number, which will satisfy the condition expressed by the equation, is called solving the equation.

21. Roots of the Equation.—The value of the unknown number, which satisfies the equation, is called **a root** of the equation and is usually found by solving the equation.



It is important for the student to remember that the two sides of the equation always represent equal numbers and that this equality must never be disturbed. An equation may be likened to a balance, having equal weights on the two pans of the balance. To keep the two sides balanced, they must be treated in the same way; whatever is done to one side must also be done to the other. If a weight is added to one side, an equal weight must be added to the other to maintain a balance. Similarly, if a weight is removed from one side, an equal weight must be removed from the other side.

In solving an equation, we therefore observe certain rules based upon truths called **axioms,** statements so self-evident that they are accepted as true.

Ax. 1. *If equal numbers are added to equal numbers, the sums are equal.*

Ax. 2. *If equal numbers are subtracted from equal numbers, the remainders are equal.*

Ax. 3. *If equal numbers are multiplied by equal numbers, the products are equal.*

Ax. 4. *If equal numbers are divided by equal numbers, the quotients are equal.*

Ax. 5. *If two numbers are equal to the same number, they are equal to each other.*

22. Transposition of Terms.—In solving equations, it becomes necessary to transfer terms from one side of the equation to the other. This is called **transposing the terms.**

1. Solve the equation, $3x + 2 = 11$

To find the value of x, which will satisfy this equation, the term containing x should be kept by itself on one side of the equation. Therefore we must transpose the term 2 to the other side. According to axiom 2, we may subtract equal numbers from each side of the equation. Subtracting 2 from each side, we have,

$$3x + 2 - 2 = 11 - 2$$

Therefore $$3x = 9$$

By Axiom 4, we may divide each side of the equation by the same number.

Dividing each side by 3, we have, $x = 3$.

2. Solve the equation, $x - b = a$

As in the preceding problem, we must transpose the term $-b$ to the other side of the equation. By Axiom 1, equal numbers may be added to each side of the equation.

Adding b to each side, we get

$$x - b + b = a + b$$

Therefore $$x = a + b.$$

In the first problem, we transposed the term 2 from the left to the right side, by subtracting 2 from each side. The effect would have been the same if we had transferred the 2 to the right side of the equation and changed its sign to minus. Similarly, in Problem 2, instead of adding b to each side, we could have transposed the term $-b$ to the right side of the equation by changing its sign to plus.

Hence we have the general **Rule for Transposition:**

Any term may be transposed from one side of the equation to the other, by changing its sign.

3. Solve the equation, $x - c = b - c$.

Transposing $-c$ to the right side of the equation,

$$x = b - c + c$$

$-c$ and $+c$, cancelling we have,

$$x = b$$

The effect is the same if we simply cancel the $-c$ on each side of the equation.

4. Solve the equation, $y + a = c + a$.

Transposing a to the right,

$$y = c + a - a$$

$+a$ and $-a$, cancel, hence

$$y = c$$

We see from Problems 3 and 4, that

Equal terms occurring on both sides of an equation may be cancelled.

If we transpose every term of the equation

$$a - x = b - c$$

we have, $c - b = x - a$

which may also be written, $x - a = c - b$

This shows that,

The sign of every term of an equation may be changed without disturbing the equality.

23. Numerical Equations.—An equation, in which all the known numbers are expressed in Arabic numerals, is called a numerical equation.

SOLUTION OF SIMPLE NUMERICAL EQUATIONS

1. Solve, $2x - 5 = 10 - 3x$

Transposing -5 to the right side and $-3x$ to the left side,

$$2x + 3x = 10 + 5$$

Combining like terms, $5x = 15$

Divide the equation by 5, $x = 3$

2. Solve the equation,

$$7 - 21(x + 3) = 13 - 15(2x - 5)$$

Multiplying the compound factors by the simple factors,

$$7 - (21x + 63) = 13 - (30x - 75)$$

Removing the parentheses,

$$7 - 21x - 63 = 13 - 30x + 75$$

Transposing all terms containing the unknown number x to the left and the known numbers to the right,

$$-21x + 30x = 13 + 75 - 7 + 63$$

Combining like terms, $9x = 144$

Divide by 9, $x = 16$

From the above solutions, we have the general **Rule** for the **Solution of a Simple Numerical Equation.**

Simplify the equation, transpose all the unknown terms to the left, the known terms to the right. Combine like terms and finally divide both sides of the equation by the coefficient of the unknown number (x or any other letter representing the unknown number.)

Note.—The word *simplify* in Algebra should be interpreted as *perform the indicated operations.* In the solution of the two problems above, it will be seen that the original forms of the equations contained indicated operations of addition, subtraction and multiplication. These operations were performed before the terms could be transposed.

VERIFICATION

24. If, when an equation has been solved, the value of the unknown is substituted in the *original equation* and this equation is then reduced to an identity, the root of the equation is said to be **verified.**

Thus in problem (1, we found $x = 3$.
Substituting this value for x in the original equation

$$2x - 5 = 10 - 3x$$

we have, $2 \times 3 - 5 = 10 - 3 \times 3$

or $6 - 5 = 10 - 9$

then, $1 = 1$

and the value of x has been verified.

EXERCISE 6

Solve and verify:—

1. $4x + 5 = 3x + 7$.
2. $10x - 7 = 9x + 8$.
3. $7x - 10 - 6x = 0$.
4. $9x + 1 = 3x + 5$.
5. $7x + 10 - 4x = 40$.
6. $5x - (2x + 3) = 12$.
7. $3x - (7 - x) = 61$.
8. $17x + 5(2 - 3x) = 18$.
9. $17 - 3(x + 11) = -7x$.
10. $5(x - 4) = 4(x - 3)$.

11. $3(x - 6) - 2(4 - x) = 0.$
12. $6(x + 5) - 12 = 3(3x - 1) + 4x.$
13. $22 - 5(3 - 2x) = x - 4(x + 8).$
14. $8(x - 7) - 6(x - 5) = 5(x - 4) - 4(x - 3).$
15. $5(x - 3) - 7(6 - x) + 29 = 56 - 3(10 - x).$

STATEMENT OF PROBLEMS

25. The one great object of Algebra is to simplify the solution of problems. It is very important that the student should learn to translate the statements of a problem into algebraic language. The following exercise is designed to give the student facility of translation before considering the actual solution of problems.

EXERCISE 7

1. Express in algebraic symbols: x increased by y; r diminished by s; a multiplied by b; v divided by t; the square of z; the cube root of v; one half of a multiplied by the square of t.

2. If one part of 30 is 12, what is the other part?

3. If one part of 25 is x, what is the other part?

4. If one part of x is a, what is the other part?

5. If the sum of two numbers is 25, and one of them is 10 what is the other?

6. If the sum of two numbers is 40 and one of them x, what is the other?

7. If the sum of two numbers is s and one of them is a, what is the other?

8. If the difference between two numbers is 8, and the smaller number is 10, what is the larger number?

9. If the difference between two numbers is d, and the smaller number is a, what is the larger number?

10. If the difference of two numbers is 5, and the larger number is x, what is the smaller number?

11. By how much does 20 exceed 15?

12. What is the excess of $7x$ over $3x$?

13. Write the excess of $5x$ over $x + 2$.

Note.—As $x + 2$ must be treated as a single number, we employ one of the signs of aggregation to indicate this fact. Thus, the excess of $5x$ over $x + 2$ is found by subtracting the number $x + 2$ from the number $5x$, and is $5x - (x + 2)$.

14. Write the excess of $3x + 12$ over $15 - x$.

15. Express in cents, a quarters and b dimes.

16. Express in cents, a dollars, b half dollars and c cents.

17. John is x years old to-day. How old was he 5 years ago? How old will he be in 5 years from now?

18. William is a years old to-day. How old was he b years ago? How old will he be c years hence?

19. A farmer can plough 2 acres in one day. How many acres can he plough in 3 days? In c days?

20. If a farmer can plough 2 acres in one day, how long will it take him to plough x acres?

21. If a man walks b miles per hour, how many miles will he walk in a hours?

22. If a man walks a miles per hour, how long will it take him to walk 50 miles? x miles?

23. What is the divisor if the dividend is 20 and the quotient 4? If the dividend is d and the quotient q?

24. What is the dividend if the divisor is 6, the quotient 5, and the remainder 2? If the divisor is d, the quotient q and the remainder r?

25. What is the divisor, if the dividend is 40, the quotient 3 and the remainder 2? If the dividend is d, the quotient q and the remainder r?

26. A man has a dollars. If he spends b quarters and c dimes, how many cents has he left?

27. If A has m dollars and B has 10 dollars less than A, and C has twice as much as A and B, how do you express the number of dollars B has and C has?

28. If x is an integer (whole number), what will represent the next higher number? What is the next lower?

29. Write three successive odd integers, the lowest of which is a.

30. If the floor of a room is x feet wide and y feet long, how many square yards of carpet are required to cover it?

SOLUTION OF PROBLEMS

26. The first step, in the solution of a problem, is to read it carefully and grasp its meaning fully.

Secondly, select the *unknown number* and represent it by some letter as x or an initial which keeps its meaning before the mind.

Thirdly, the conditions given in the problem are translated into algebraic language, and two equal algebraic numbers are found to form the two sides of the equation.

Finally, solve the equation for the unknown number and state the results thus found.

1. The sum of two numbers is 84 and the larger is six times the smaller. What are the numbers?

Let x = the smaller number

Then $6x$ = six times the smaller number

And $x + 6x$ = the sum of the two numbers

But the sum of the two numbers is given equal to 84

Therefore $x + 6x = 84$

Combining like terms, $7x = 84$

Dividing both sides of the equation by 7, the coefficient of x,

$$x = 12$$

Therefore the smaller number is 12, and the larger number is six times the smaller or 72.

2. Find three consecutive odd numbers whose sum is 21.

Let x = the smallest number

Then $x + 2$ and $x + 4$ will represent the other two. But their sum is equal to 21.

Therefore, $x + x + 2 + x + 4 = 21$

Transposing, $x + x + x = 21 - 2 - 4$

Combining like terms, $3x = 15$

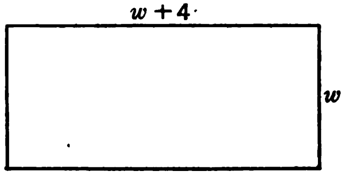
Dividing by 3, $x = 5$

Then $x + 2 = 7$

And $x + 4 = 9$

Therefore the three odd numbers are 5, 7, and 9.

3. A rectangle is 4 inches longer than it is wide. Find its length and width if the perimeter is 40 inches. (Perimeter means the sum of the sides).



Let w = the number of inches in the width
then $w + 4$ = the number of inches in the length

The number of inches in the perimeter then is, $2w + 2(w + 4)$. But the perimeter is 40 inches.

Therefore $2w + 2(w + 4) = 40$

Multiplying the compound by the simple factor, we have

$2w + (2w + 8) = 40$

Removing the parenthesis, $2w + 2w + 8 = 40$

Transposing, $2w + 2w = 40 - 8$

Combining like terms, $4w = 32$

Dividing by 4, $w = 8$

And, $w + 4 = 12$

That is, the width is 8 inches, and the length 12 inches.

4. The number of representatives and senators together in the United States Congress is 531. The number of representatives is 51 more than 4 times the number of senators. Find the number of each.

Let s = the number of senators
Then, $4s + 51$ = the number of representatives

But the total number is equal to 531.

Therefore, $s + 4s + 51 = 531$

Transposing, $s + 4s = 531 - 51$

Combining like terms, $5s = 480$

Dividing by 5, $s = 96$

And, $4s + 51 = 4 \times 96 + 51 = 384 + 51 = 435$

Therefore the number of senators is 96, and the number of representatives is 435.

EXERCISE 8

Note.—Do not let x, or whatever letter may be used, represent money, age, weight, length and so on, but the *number of units* in these quantities. See the problems explained above.

1. One number is 4 times another, and their difference is 12. What are the numbers?

2. Three times a certain number is equal to the number increased by 12. Find the number.

3. A and B together have $50, and B has four times as much as A. How much has each of them?

4. Three trees together bear 70 bushels of apples; the second tree bears six times as much as the first, and the third only half as much as the second. How many bushels does each tree bear?

5. Find three numbers such that the second is 8 times the first and the third is 3 times the second. The third number less the second equals 48.

6. John is 3 times as old as Henry, and the difference of their ages is 14 years. How old is each?

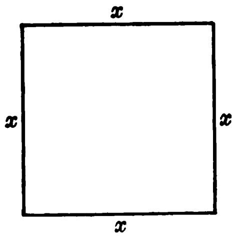
7. Thirty yards of cloth and forty yards of silk together cost $66; the silk costs twice as much per yard as the cloth. Find the cost per yard of each of them.

8. Find three consecutive integers whose sum is 33.

9. The area of Illinois is 6750 square miles more than 10 times that of Connecticut. The sum of their areas is 61,640 square miles. Find the area of each state.

Kinney

10. Find the side of a square whose perimeter is 44 inches.



11. Find four consecutive odd integers such that 7 times the first equals 5 times the last.

12. A rectangle is twice as long as it is wide. Find

its dimensions if the perimeter exceeds its length by 36 feet.

13. If $5x + 4$ stands for 39, for what number will $x - 2$ stand?

14. A's age exceeds B's age by 14 years. Eight years ago A was three times as old as B. Find the present age of each.

15. The sum of the ages of A and B is 36 years, and six years hence A's age will be three times that of B. Find their present ages.

16. A coal dealer estimated that his supply of coal would last six weeks. He sold, on an average, 10 tons more per day than he expected. It lasted him 4 weeks. How much did he have?

17. Divide $135 among three persons so that the first shall receive three times as much as the second and the second twice as much as the third.

18. A man wishes to divide $99 into five parts, so that the first part shall exceed the second by $3, be less than the third part by $10, greater than the fourth part by $9 and less than the fifth part by $16. Find the parts.

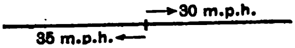
19. A farmer estimated that his supply of feed for his 50 cows would last only 12 weeks. How many cows must he sell in order that the supply may last 20 weeks?

20. A contractor agrees to complete a certain amount of work in a given time. According to the contract, he is to receive $20 for each day's work during the given time, but is to forfeit $10 for each day taken beyond that time. If the total amount received for 25 days' work was $350, find the number of days allowed by the contract.

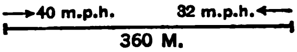
21. A factory employs 250 workers, consisting of men and women. If the weekly pay of each man is $18 and that of each women $12, find the number of each, if the weekly payroll amounts to $3900.

22. Two trains start from the same station at the same time and travel in opposite directions, one at the rate

of 30 miles an hour, the other at 35 miles an hour. In how many hours will they be 260 miles apart?



23. Two trains start at the same time from different stations 360 miles apart. One travels at the rate of 40 miles an hour and the other at the rate of 32 miles an hour. How far does the slower train travel before meeting the faster train?



24. Two hours after a train left a certain station, a second train started out and overtook the first train in 5 hours. To do this, the second train had to run 10 miles an hour faster than the first train. What was the speed of each?

25. An excursion boat makes the up-river trip at 8 miles per hour and the down-river trip at 12 miles per hour, the whole time being 10 hours. Find the distance.

26. A tank holding 7200 gallons has three pipes. The first lets in 6 gallons per minute, the second 8 gallons, and the third 10 gallons. In how many minutes will the tank be filled?

27. The front and rear wheels of a carriage are 12 and 14 feet respectively in circumference. How many feet will the carriage have passed over when the front wheel has made 200 revolutions more than the rear wheel?

28. Twenty persons rented a hall, but four being unable to pay, each of the others had to pay $1 more than his share. Find the rental of the hall.

29. A stream runs at the rate of 2 miles an hour. A man swims a certain distance up the river in 3 hours and the same distance down in 1 hr. Find his rate of swimming in still water.

30. A man having only half dollars and quarters wishes to give some persons 75 cents each, but found that he had not money enough by $1.25. He therefore gave them

50 cents each and had \$1.50 left. How many persons were there?

31. A grocer has two kinds of coffee, one worth 40 ¢ a pound and the other 30 ¢ a pound. He makes a mixture from these of 100 pounds, worth 32 ¢ a pound. How many pounds of each kind does he take?

32. A sum of money is divided among three persons, A, B, and C, so that A and B together have \$10, A and C \$12, and B and C \$14. How much has each?

33. A merchant sold 3 equal lots of cloth for \$56. For the first lot he received \$2 per yard, for the second \$2.75, for the third \$3.25. How many yards of each kind did he sell?

34. A tree 70 feet high was broken so that the part broken off was four times the length of the part left standing. Find the length of each part.

35. Twice the excess of a certain number over 5 is equal to the number increased by 5. Find the number.

36. Twenty-one times a number exceeds 75 by as much as four times the number is less than 75. Find the number.

37. A has \$20.25 and B has \$3.75. How much must A give to B in order that he may have just three times as much as B?

REVIEW EXERCISE III

1. Solve for x: $4x^3 - 4(x^3 - x^2 + x - 2) = 4x^2$.

2. A man paid yearly a certain amount for taxes and twice as much for improvements and received for rent three times as much as he paid for improvements. If his net gain per year was \$300, what were his taxes?

3. A speculator who doubled his money by a fortunate investment, afterward lost \$600, but he still had \$400 more than the original sum. How much did he invest?

4. Solve and verify: $7(3x - 6) + 5(x - 3) + 4(17 - x) = 77$.

5. A's age exceeds B's by 14 years. Eight years ago A was 3 times as old as B. Find the present age of each.

6. Divide \$135 among A, B, and C, so that A's share is 3 times B's and B's is twice C's.

7. Solve and verify, $2(x+1) - 3(x+1) + 9(x+1) + 18 = 7(x+1)$.

8. A man has \$4.75 in quarters and dimes, and he has 5 more quarters than dimes. How many coins of each kind has he?

9. A teacher proposes 16 problems to a pupil. The latter is to receive 5 marks in his favor for each problem solved, and 3 marks against him for each problem not solved. If the marks in his favor exceed those against him by 32, how many problems will he have solved?

10. A man leaves his estate amounting to \$7500, to be divided among his wife, two sons, and three daughters. A son was to have twice as much as a daughter, and the wife \$500 more than all the children together. Find the share of each.

11. A man and two helpers together earned \$7.50 a day. How much did each earn per day if the man earned 4 times as much as each helper?

12. Solve and verify: $6x + 4(4x+2) + 3(2x+7) = 85$.

13. A bicyclist averaging 12 miles an hour is 52 miles ahead of an automobile running 20 miles an hour. How soon will the automobile overtake him?

14. A rectangular shed sheltering an airship was 272 feet in perimeter. If twice its length was 26 feet more than four times its width, find its dimensions.

15. Forty stamps, two's and three's, cost 95 ¢. How many were there of each?

16. A dealer paid \$185 for some boxes of candles, at \$9 for some, and at \$6.50 for the others. The number of the latter was 2 less than twice the number of the former. How many of each kind did he buy?

17. Divide \$152 among 5 men, 7 women, and 30 children, giving to each man \$4 more than to each woman, and to each woman three times as much as to each child.

18. A man went to a bank with a check for $36 and asked to have it cashed in half dollars, quarters, dimes, and nickels, of each the same number. What was the number?

19. Solve: $5(3x - 5) - 2(3x - 5) = 44 + 8(5 - 3x)$.

20. Solve and verify: $80 - 6(4x + 3) = 7x - 3(6x + 1)$.

CHAPTER IV

ADDITION AND SUBTRACTION

27. Algebraic Addition and Subtraction differ from the similar operations in Arithmetic only in that algebraic as well as arithmetical numbers are included, that is, negative, as well as positive, numbers.

To indicate the addition or subtraction of algebraic numbers, a single number is often inclosed in a parenthesis.

Thus $4 + (-3)$ indicates that -3 is to be added to 4, the parenthesis being used to avoid confusion between the sign of operation and the minus sign belonging to 3.

When numbers have the same sign, they are said to have *like signs*, and when their signs are different, they have *unlike signs.*

It is evident that when two numbers have like signs, their sum is found by simple arithmetical addition, giving to the sum, the sign of the numbers.

But, since positive and negative numbers are opposite in kind, they will oppose each other when combined by addition, that is, the addition of a negative number will have the effect of diminishing the positive number.

Thus,

$$7 + (-3) = 7 - 3 = 4$$
$$3 + (-7) = 3 - 7 = -4$$

Or, in general,

$$\begin{aligned} a + (+b) &= a + b \\ a + (-b) &= a - b \\ -a + (+b) &= -a + b \\ -a + (-b) &= -a - b \end{aligned}$$

We therefore have the following

Rule for Adding Two Algebraic Numbers.

1. *If the numbers have like signs, find the sum of their*

absolute values (values without regard to their signs) and prefix the common sign to the result.

2. *If the numbers have unlike signs, find the difference between their absolute values, and prefix the sign of the greater to the result.*

In either case, the result is called the *Algebraic Sum.*

Where more than two numbers are involved, it is convenient to add two of them, then add their sum to the third and so on; or find the sum of all the positive numbers and the sum of all the negative numbers and add these sums as any two algebraic numbers are added.

EXERCISE 9

Add:

1. +9 −7	2. −9 +7	3. −9 −9	4. −13 + 4	5. −17 − 8
6. +3 −4 +7	7. +3 −7 −5	8. −5 +4 −7	9. −11 − 4 − 8	10. +21 − 3 −10
11. +11 +15 −12	12. −11 −15 +10	13. −17 +23 − 6	14. −23 −17 +20	15. +17 −34 −15
16. +15 +19 −13 −11	17. +15 −19 −22 −13	18. −15 +19 −22 +13	19. +22 +18 −11 +17	20. −44 +16 −22 +30

ADDITION OF SIMILAR MONOMIALS

28. Since different letters usually stand for different numbers, it is evident that they cannot be combined by addition. Thus, we cannot add a to b, in the ordinary arithmetical way; all we can do is to indicate the addition by connecting them with a plus sign, as, $a + b$ On the other hand, where the letters are the same, as $2a$ plus $4a$, their sum is found by simply adding their coefficients, because, twice a certain number plus four times that number will equal six times that number. If these letters have different exponents, as $2a^2$ plus $4a^3$, they cannot be combined by actual addition, since a^2 and a^3 will stand for unequal numbers, just as the different letters a and b represent different numbers.

It follows therefore, that *we can only add similar terms*, that is, terms having the same letters, with equal exponents, although their coefficients may be different.

1. Add: $2x$, $4x$, $-3x$, $-5x$, $7x$.

Adding the coefficients,

$$2+4+(-3)+(-5)+7=+5.$$

Therefore the sum is $5x$.

2. Add: $4a^2b^3$, $-6a^2b^3$, $-9a^2b^3$, $-3a^2b^3$.

Adding the coefficients,

$$4+(-6)+(-9)+(-3)=-14.$$

Hence, the sum is, $-14a^2b^3$.

Rule for Finding the Sum of Similar Monomials. *Add the coefficients and annex to the sum the common letters with their proper exponents.*

EXERCISE 10

Find the sum of:

1.	2.	3.	4.	5.
$5a$	$-7a$	$4xy$	$-3a^2b$	$8mn^2$
$2a$	$5a$	$-2xy$	$-5a^2b$	$-9mn^2$
$-3a$	$2a$	$7xy$	$7a^2b$	$2mn^2$
$-a$	$-3a$	$-9xy$	$4a^2b$	$-7mn^2$

Find the algebraic sum of:

6. $9c$, $-3c$, $-5c$, $8c$.
7. $4uv$, $6uv$, $-9uv$, $-3uv$.
8. $5at$, $-7at$, $-3at$, $9at$.
9. $-7z^3$, $5z^3$, $6z^3$, $-10z^3$.
10. $3xy^2$, $-7xy^2$, $-5xy^2$, $11xy^2$, $-9xy^2$.
11. $5abc+4abc-7abc+9abc-2abc$.
12. $-8x^2y^2+3x^2y^2-5x^2y^2-7x^2y^2+11x^2y^2$.
13. $7\sqrt{x}-3\sqrt{x}+8\sqrt{x}+\sqrt{x}-9\sqrt{x}$.
14. $-10at^2+11at^2+4at^2-16at^2$.

15. $3\sqrt[3]{a} + 7\sqrt[3]{a} - 5\sqrt[3]{a} - 4\sqrt[3]{a} + \sqrt[3]{a}$.
16. $\frac{1}{2}b - \frac{1}{3}b + \frac{1}{4}b - \frac{1}{6}b$.
17. $\frac{2}{3}a - \frac{1}{4}a + \frac{1}{12}a - \frac{5}{6}a + a$.
18. $3c^2 - \frac{1}{3}c^2 - \frac{4}{8}c^2 + \frac{5}{6}c^2 - c^2$.
19. $13xyz + 5xyz - 17xyz - \frac{1}{6}xyz$.
20. $3(a + b) - 2(a + b) - 4(a + b) + 7(a + b) - 9(a + b)$.

SUBTRACTION OF ALGEBRAIC NUMBERS

29. To subtract one algebraic number from another, we may apply the rule for removing parentheses.

Thus $5 - (3) = 5 - 3 = 2$

And $5 - (-3) = 5 + 3 = 8$

When a minus sign precedes, a parenthesis may be removed by changing the signs of the terms in the parenthesis.

Hence, the **Rule for subtracting one algebraic number from another:**

Change the sign of the subtrahend and add the resulting number to the minuend.

This rule can easily be explained if we consider concrete numbers. Suppose a man's assets to be called positive and his debts negative. If a man has \$10 and owes \$3, he is actually worth only \$7. If, however, his debt is removed, he will be worth \$10, that is, the subtraction of his debt, the minus \$3, is equivalent to an actual addition of plus \$3 to his assets.

EXERCISE 11

Perform the indicated subtraction:

1.	2.	3.	4.	5.	6.
10	10	−10	−10	4	4
4	−4	4	−4	10	−10
7.	8.	9.	10.	11.	12.
−4	−4	2	−2	−2	13
10	−10	−7	−7	7	−5

SUBTRACTION OF SIMILAR MONOMIALS

30. Since any monomial represents some algebraic num-r, it is evident that the rule used in subtracting one algebraic number from another, is also used in the subtraction of monomials.

That is, *to subtract one monomial from another, change the sign of the subtrahend and then proceed as in addition.*

Since only similar monomials can actually be combined by addition, it follows, that only similar monomials can be combined by subtraction. If the monomials are not similar, their subtraction can only be indicated by connecting them by the sign of subtraction.

Thus, we can actually subtract $3x$ from $7x$, giving us the remainder $4x$, but, the subtraction of $3a$ from $7b$ can only be indicated, the difference being written, $7b - 3a$.

EXERCISE 12

Perform the indicated subtractions:

1.	$\begin{array}{r}10a\\ 4a\\ \hline\end{array}$	2.	$\begin{array}{r}10a\\ -4a\\ \hline\end{array}$	3.	$\begin{array}{r}-10a\\ 4a\\ \hline\end{array}$	4.	$\begin{array}{r}-10a\\ -4a\\ \hline\end{array}$
5.	$\begin{array}{r}3xy\\ 4xy\\ \hline\end{array}$	6.	$\begin{array}{r}11mn^2\\ -5mn^2\\ \hline\end{array}$	7.	$\begin{array}{r}5x^2y\\ -7x^2y\\ \hline\end{array}$	8.	$\begin{array}{r}-22a^2b^2\\ 13a^2b^2\\ \hline\end{array}$
9.	$\begin{array}{r}-15abc\\ -21abc\\ \hline\end{array}$	10.	$\begin{array}{r}37yz\\ -23yz\\ \hline\end{array}$	11.	$\begin{array}{r}-39yz\\ 26yz\\ \hline\end{array}$	12.	$\begin{array}{r}44abc\\ -\quad abc\\ \hline\end{array}$

31. An **Integral Expression** is an algebraic expression containing no letters in the denominators of any of its terms. Thus, $3x^2 + 5x - 2$ and $ut + \frac{1}{2}at^2$ are integral expressions.

Note carefully, that while in arithmetic, a number, to be integral must contain no fractions, in algebraic integral expressions, we may have numerical fractions, as in the second expression given above. Also note, that the name, integral expression, refers only to the form of the expression, since even an integral expression may have a *fractional value*. Thus, the integral expression, $2x - 3y$, may have a fractional value if $x = \frac{1}{3}$ and $y = \frac{1}{5}$, in which case $2x - 3y = \frac{2}{3} - \frac{3}{5} = \frac{10}{15} - \frac{9}{15} = \frac{1}{15}$. On the other hand, a fractional expression may have an integral value. As, $\frac{4a}{3b}$, if $a = \frac{1}{4}$ and $b = \frac{1}{6}$, when

$$\frac{4a}{3b} = \frac{4}{4} \div \frac{3}{6} = 1 \div \frac{1}{2} = 1 \times \frac{2}{1} = 2$$

ADDITION OF INTEGRAL COMPOUND EXPRESSIONS

32. The addition of two integral compound expressions may be indicated by connecting the two expressions by the sign of addition. Thus, to indicate the addition of $a + b - c$ and $x - y + z$, we write $a + b - c + (x - y + z)$. Removing the parenthesis, we get $a + b - c + x - y + z$ as the sum. If this sum contains any similar monomials, they are combined in the usual way by algebraically adding their coefficients. Thus, the sum of $a + 2b - 3c$ and $2a - b + 2c$ equals $a + 2b - 3c + (2a - b + 2c) = a + 2b - 3c + 2a - b + 2c = 3a + b - c$.

This operation is most conveniently performed by placing the similar terms in the same column.

As,

$$\begin{array}{rrr} a & +\ 2b & -\ 3c \\ 2a & -\ \ b & +\ 2c \\ \hline 3a & +\ \ b & -\ \ c \end{array} \text{ the sum.}$$

This same method is used when more than two expressions are to be added.

Thus, add $8mn^2 + 3x^2y^3 + 5a$, $7x^2y^3 + 3mn^2 - 7a$, $2mn^2 - 7x^2y^3 + 3a$.

Arranging like terms in the same column,

$$\begin{array}{rrr} 8mn^2 & +\ 3x^2y^3 & +\ 5a \\ 3mn^2 & +\ 7x^2y^3 & -\ 7a \\ 2mn^2 & -\ 17x^2y^3 & +\ 3a \\ \hline 13mn^2 & -\ 7x^2y^3 & +\ a \end{array}$$

Adding,

EXERCISE 13

Add:—

1. $x + y$; $x - y$.
2. $7b - 2c + 3d$; $-8b + 2c - 11d$.
3. $6x - 4y + 3t - 5z$; $y + 2t - z$; $2x + 5y - 4t$.
4. $7a + 5c + 3x^2y$; $5x^2y - 7b - 3a$; $2c - 4x^2y + 3b$.
5. $2abc + 3a^2 - 4b^2 + 6ac + 5bc$; $4a^2 - 3bc + 2b^2$; $5ac - 2b^2 + 6ac - 2b^2$; $5b^2 - 3ac + abc - bc - a^2$.
6. $2y^3 - z^2 + 9$; $z^2 + w^2 + 10$; $y^3 + w^3 - 4$.
7. $3m^3 - 7 + x^2$; $4x + 5m^3$; $6m^3 - 2x^2 + 3x + 9$.
8. $5p^2q - 7pq^2 + 5p^3 - 4q^3$; $3p^3 - 4p^2q + 2pq^2 - 5q^3$; $6pq^2 - 7p^3 + 4p^2q + 8q^3$; $3q^3 - 4pq^2 - 7p^2q + 6p^3$.

9. $8x^3 + 7x - 12$; $-5x^2 + 11$; $-6x^3 + x^2 - 3x$; $4x^2 - 3x + 5$.

10. $3x^2 + 2xy + 5xy^2 + y^3$; $3xy - 4xy^2 - 2y^3 + x^3$; $3xy^2 + 2x^2 - x^3 + xy$.

11. $3a^2 - ab - ac - 4bc - 2b^2 + c^2$; $5c^2 - 3bc - ab$; $-ab + ac - 5bc$; $-2a^2 - 6bc + b^2 + 3c^2$.

12. $x^4 - 3x^3 + 6x^2 - 5x + 3$; $2x^3 - x^4 + 5x^2 - 6x - 5$; $-3x^4 + x^2 - x$; $5x^2 - x^4 + 6x + 2$.

13. $6xy^2 - 8y^3 - 2x^2y - x^3$; $12xy^2 + 4x^3 - 10xz^2 + y^3$; $-xz^2 + x^2y + 3xy^2 - 2y^2$; $8x^3 - 6xz^2 - 2y^2 + 3z^2 - xy^2$.

14. $2a^4 - 3a^3 + 4a^2 - 5a + 6$; $4a^3 - 3a^4 + 6a - 8$; $6a^4 + 7 - 3a^3$; $5a^3 - a^4 + a^2 + 7a + 9$.

15. $u^3 + 2uv^2 - u^2v - v^3$; $2u^3 + 3u^2v + 4uv^2 + 6v^3$; $6v^3 - 5uv^2 - u^3$.

16. $r^2 - 3r^4s + 2r^3s^2 - s^3$; $4r^4s - 5r^2 + 6r^3s^2 + 4s^3$; $10s^3 + 5r^2 - 6r^4s$; $6s^3 - r^4s$.

17. $m^5 + 3m^3n^2 - 4mn^4$; $4m^4n - 2m^2n^3 + n^5$; $-2m^5 + 6m^4n - 2n^5$; $5m^4n - 2m^3n^2 + 6m^2n^3 - 2mn^4$.

18. $x^2y - 2xy^2 - 4y^3 - 1$; $2 + xy^2 + y^3$; $6 - 4x^2y + xy^2$; $-x^2y - 7 + 3xy^2$; $5y^3 + 2 - 3x^2y - xy^2$.

19. $a^3 - b^3$; $a^2 + ab + b^2$; $2a^3 - 3a^2b + 3ab^2 - b^3$; $4a^2b + 6ab^2$; $-a^3 + 3a^2b - 3ab^2 - b^3$; $4ab^2 + 6a^2b + a^3 + b^3$.

20. $2x^2 + 3y^2 + 4z^2 - 6xy$; $x^2 + 5xz - 3yz$; $y^2 - z^2 + 2xyz$; $6xy + 4yz - 3xyz + z^2$; $-2yz - x^2 - y^2 + z^2$.

SUBTRACTION OF INTEGRAL COMPOUND EXPRESSIONS

33. To subtract one compound expression from another, we indicate the subtraction by placing a minus sign between the minuend and subtrahend.

Thus, the subtraction of $a + b - c$ from $a + b + c$ is written, $a + b + c - (a + b - c)$.

Removing the parenthesis, we have,

$$a + b + c - a - b + c.$$

Combining like terms, the remainder becomes $2c$.

Hence the **Rule for subtracting one compound expression from another:**

Change the signs of the subtrahend and add.

For convenience, we follow the method used in addition, writing like terms under each other.

$$\begin{array}{lrrrr} \text{Thus, from} & a^2 + & b^2 & & +c^2 \\ \text{Subtract} & 2a^2 - & b^2 & -ab & \\ \cline{2-5} \text{Changing signs and adding,} & -a^2 + & 2b^2 & +ab & +c^2 \end{array}$$

The changes in sign should be made mentally.

EXERCISE 14

Perform the indicated subtractions:

1. From $5a - 3b + 4c$ take $3a + b - 2c$.
2. From $2a + b - c$ take $2a - 2b - 2c$.
3. From $a^2 - b^2$ take $a^2 + b^2$.
4. From $a + b$ take $a - b$.
5. From $2xy + y^2$ take $xy - x^2$.
6. From $a^2 + ab + b^2$ take $a^2 - ab + b^2$.
7. From a^3 take $b^3 - a^3$.
8. From b^3 take $b^3 - a^3$.
9. From $5a^2 - 3ac + b^2$ take $2b^2 + 5ac - a^2$.
10. From $2x^4 - 3x^3 + 4x^2 - 5x - 10$ take $2x^3 - x^4 + 5 - 2x$.

If $A = 10a^2 - 4ab - 3b^2$ $\quad B = 6a^2 - 7ab - 3b^2$.
$C = 8a^2 - 6ab + 4b^2$ $\quad D = 2a^2 + 6ab + 2b^2$.
Find the expression for $A - B - C + D$.

This problem means that the expressions represented by A, B, C, and D are to be combined by addition or subtraction, according to the signs that connect them. This can most conveniently be done in a single addition by remembering that the subtrahend must have its signs changed before being added to the minuend.

$$\begin{array}{lrrrrl} \text{Therefore,} & A = & 10a^2 - & 4ab & -3b^2 & \\ & -B = & -\ 6a^2 + & 7ab & +3b^2 & \\ & -C = & -\ 8a^2 + & 6ab & -4b^2 & \\ & D = & 2a^2 + & 6ab & +2b^2 & \\ \cline{3-5} \text{Adding,} & & -\ 2a^2 + & 15ab & -2b^2 & = A - B - C + D \end{array}$$

Using the same values for A, B, C, D, find the expression for:

11. $A + C + D - B$. 12. $A + D - C + B$.
13. $B - A + D - C$. 14. $B - A - C - D$.
15. $D - A - B - C$. 16. $C - A + B - D$.

REMOVAL OF PARENTHESES

34. Expressions may occur, having one sign of aggregation inside of another. For this purpose we use the different signs, parenthesis ( ), bracket [ ], brace { }, and vinculum $\overline{}$. The rules for removing a parenthesis are applied here and it is usual to remove one sign of aggregation at a time, beginning with the *innermost.*

Thus,

$$18 + a - \{6a - [7a - (8a - \overline{5a - 3})]\}$$

Removing the vinculum first,

$$= 18 + a - \{6a - [7a - (8a - 5a + 3)]\}$$

Removing the parenthesis,

$$= 18 + a - \{6a - [7a - 8a + 5a - 3]\}$$

Removing the bracket,

$$= 18 + a - \{6a - 7a + 8a - 5a + 3\}$$

Removing the brace,

$$= 18 + a - 6a + 7a - 8a + 5a - 3$$

Collecting like terms,

$$= 15 - a$$

EXERCISE 15

Simplify:

1. $x - y - 3z - (3x + [2y - z] + y)$.
2. $10t + (2t - 1) - [1 - (3 + t)]$.
3. $4x - (8 - \overline{4 + 5t} + 7)$.
4. $17 - (2 - 7a - \overline{4b + 11ab}) - (-6b - 13by + 17ab)$.
5. $x - \{8x - (3y - 4x) + (4x - 2y)\}$.
6. $a - \{ - (a - b) + (3a - 2b) - a\}$.
7. $2x - (x - 1) - [x - (2x + 1)]$.
8. $- \{a - [a - (b - c) - (a + b + c]\}$.
9. $-\{ - [- (- \overline{a - b} - c - d)]\}$.
10. $- \{ + [- (- \overline{a + b} + c - d) + a] - b\}$.
11. $5a + \{-3a + [3a - (2a + \overline{a - b})] + a\}$.
12. $x - \{5y - [x - (2z + 3y) + x - (y+2x-z)]\}$.

INSERTION OF PARENTHESES

35. The insertion of various terms in a parenthesis is the reverse of the removal of a parenthesis and hence the same rules are observed.

Since $a + (b + c) = a + b + c \therefore a + b + c = a + (b + c)$
$a + (b - c) = a + b - c \therefore a + b - c = a + (b - c)$
$a - (b + c) = a - b - c \therefore a - b - c = a - (b + c)$
$a - (b - c) = a - b + c \therefore a - b + c = a - (b - c)$

We have the following rules:

1. *Any number of terms may be enclosed in a parenthesis, preceded by a plus sign, without changing the signs of these terms.*

2. *Any number of terms may be enclosed in a parenthesis, preceded by a minus sign, by changing the signs of these terms.*

Thus, $x - y - z + 3x - y + 2z = x - y - z + (3x - y + 2z)$
Also, $x - y - z + 3x - y + 2z = x - y - (z - 3x + y - 2z)$

EXERCISE 16

In the following expressions, enclose the last three terms in a parenthesis, preceded by a minus sign.

1. $10t + 2t - 1 - 1 + 3 - t$.
2. $4x - 8 + 5t - 4$.
3. $17 - 7a - 4b + 11ab - 13\,by$.
4. $x - 8y + 3z + u - v$.
5. $5v + 4s - u + t - x$.
6. $-a + b - c + 2e - 3f$.
7. $2a^2 + 3t^2 - \frac{1}{2}at - st + v$.
8. $4a^2 - 5b^2 + 6ab - c^2 + 2ac + 3bc$.
9. $ax + bx - ay + cy$.
10. $a^2b^2 + a^2c^2 - b^2c^2 + abc - b^2c$.

In algebraic addition, like terms are added by adding their coefficients, using the sum of these coefficients as the coefficient of the sum. Thus, $3x - 2x + 5x = 6x$.

If the coefficients are literal, and therefore cannot be actually combined by addition, this addition is indicated by enclosing these coefficients in a parenthesis. Thus, $ax + bx - x = (a + b - 1)x$.

In the following expressions, collect like terms by enclosing their coefficients in parentheses:

11. $ax + by + cz + bx - az + y =$
$ax + bx + by + y - az + cz =$
$(a + b)x + (b + 1)y - (a - c)z.$

Note that these coefficients are preferably arranged in alphabetical order and if the first of any set of coefficients is negative, a minus sign is used before the parenthesis.

12. $ax + 2ay + 4z + bx - 3y - 2bz + 2z.$
13. $2ax - 2by - 2cz - 2bx + 3cy - 5az.$
14. $3by - 3ax - 6cz + 3bx - 11cx + 2cy - cz.$
15. $5az - 4by + 3cz + 2bx - 7cx - 5cy + x - y + z.$
16. $z + by - 2az + 3cy + 2ax - 2x - 5z.$
17. $x - ay - az - acx + bcz - mx + y - z.$
18. $2bx - 6by + 4az - 4ax - 2cx - 3cy.$
19. $ax - 2ay + 4cz - 4bx - 3cy + az - 2cx + by + bz.$
20. $10ax + 10ay - 10az + 4by - 12bz - 11cx - 6cy - 3cz.$

REVIEW EXERCISE IV

1. If $M = x^2 - 2xy + y^2$, $N = 3xy - y^2$, $P = 2x^2 + xy$, and $R = x^2 + xy + y^2$, find the value of $M - \{N - 2(R - P)\}$.

2. If $x = 2a^2 - 3ab + b^2, y = b^2 - 6ab, z = a^2 + 2ab - 3b^2$, find the value of $x - (2y + 3z)$

3. Simplify: $16 - x - [7x - \{8x - (9x - \overline{3x - 6x}\}]$.

4. Simplify: $4 - \{[5y - (3 - \overline{2x - 2})] - [x + (5y - \overline{x + 3})]\}$.

5. Simplify: $x^3 - [x^2 - (1 - x)] - \{1 + [x^2 - (1 - x) + x^3]\}$.

6. Simplify: $a + 2b + (14a - 5b) - \{6a + 6b - (5a - \overline{4a - 4b})\}$.

7. Simplify: $a^2 + 5 - [2ab - \{-(7 - 3ab) - \overline{ab + 2a^2 - z}\} - (3a - z)]$.

8. Simplify: $2x - (3y - 4z) - \{2x - [3y + \overline{4z - 3y} - (4z + 2x)]\}$.

9. Simplify: $4a - [a - \{-7a - (8a - \overline{5a + 3}) - (-6a - \overline{2a - 9})\}]$.

10. Collect coefficients of like terms in:

$$ax^3 - x^2 - 4 - 4x^3 + ax^2 - px - cx^3$$

11. If $a = 4x^2 + 7xy + 6y^2$, $b = 3xy - 4x^2$, $c = 2xy - 8y^2 + x^2$, $d = 2y^2 - x^2 - 12xy$, prove that $a + b + c + d = 0$.

12. What must be added to $5a^3 - 7a^2b + 3ab^2$ in order that the sum may be, $a^3 - 2a^2b - 2ab^2 + b^3$?

13. Collect the coefficients of like terms in:

$$ax^3 - bx^2 - cx - cx^3 - dx + x^2$$

14. Collect the coefficients of like terms in:

$$3x^3 + x - 2x^2 - ax^3 - 5 + ax^2 - 2ax - cx^3 - cx^2 - cx$$

15. Simplify: $x - [-\{-(-x)\}]$.

16. If $A = 6z - 7x + 2y$, $B = 8y - 3z + 3x$, $C = 3z - 2x + 5y$, $D = 4x + 3y - 5z$, find the value of $B - \{C + (A - D)\}$.

CHAPTER V

MULTIPLICATION AND DIVISION

36. In Arithmetic, *muliplication* was defined as the process of taking one number, called the multiplicand, as many times as there are units in the other number, called the multiplier, and the result was called the product. Multiplication may then be considered an addition of a specified number of equal numbers. Thus $3 \times 7 = 7 + 7 + 7$, that is, the number 7 is taken three times. It is evident that multiplication is only a shortened addition of equal numbers.

Our definition of multiplication fails when the multiplier is a fraction, for then we divide the multiplicand into as many equal parts as there are units in the denominator and take as many of these parts as there are units in the numerator.

Thus, $\frac{3}{4} \times 8 = 2 + 2 + 2 = 6$.

Here we can see that the multiplier $\frac{3}{4}$ has the same relation to a unit as the product 6 has to the multiplicand. Hence we get a more *general definition of multiplication*, as *the process of obtaining the product from the multiplicand in the same way as the multiplier is obtained from unity.*

Algebraic multiplication differs from Arithmetic because of the use of negative as well as positive numbers.

LAW OF SIGNS IN MULTIPLICATION

37. According to our definition of multiplication, since
$+4 = +1 + 1 + 1 + 1$, adding 4 units,
$+4 \times (+6) = +6 + 6 + 6 + 6 = 24$, adding four plus sixes,
$+4 \times (-6) = -6 + (-6) + (-6) + (-6) = -24$, adding four minus sixes.

Also since $-4 = -1 - 1 - 1 - 1$, subtracting four units, $-4 \times (+6) = -6 - 6 - 6 - 6 = -24$, subtracting four sixes and $-4 \times (-6) = -(-6) - (-6) - (-6) - (-6)$ subtracting four minus sixes equals $6 + 6 + 6 + 6 = 24$.

From these multiplications it is evident that if a and b stand for any two numbers,

$$\begin{aligned} (+a) \times (+b) &= +ab \\ (+a) \times (-b) &= -ab \\ (-a) \times (+b) &= -ab \\ (-a) \times (-b) &= +ab \end{aligned}$$

That is when the multiplier and multiplicand have like signs, the product, is plus, but when the multiplier and multiplicand have unlike signs, the product is minus.

Therefore, the **Law of Signs in Multiplication.** *Like signs give plus, and unlike signs give minus.*

THE INDEX LAW IN MULTIPLICATION

38. The exponent of a letter indicates the number of equal factors of the product, thus $x^3 = x \cdot x \cdot x$.

If therefore, x^3 is multiplied by x^2, two additional equal factors are put into the product, making five in all, that is, $x^3 \times x^2 = x \cdot x \cdot x \cdot x \cdot x = x^5$.

It is evident, therefore, that when two or more factors containing the same letter are multiplied, the exponent of that letter, in the product, will be equal to the sum of the exponents of that letter in the various factors. Thus $x^3 \cdot x^2 \cdot x^4 = x^{3+2+4} = x^9$.

This is true whether the exponents are numerical or literal, that is $x^a \cdot x^b = x^{a+b}$.

Hence, **The Index Law in Multiplication.** *The exponent of a letter in the product is equal to the sum of the exponents of that letter in the factors.*

MULTIPLICATION OF MONOMIALS

39. Since the factors of a product may be written in any order,

1. $3x^2y \times 4xy^3 = 3 \times 4 \times x^2 \times x \times y \times y^3 = 12x^3y^4$.

Note that we only add the exponents of like letters.

2. $-5a^2b \times 2b^2c = -5 \times 2 \times a^2 \times b \times b^2 \times c = -10a^2b^3c$.

3. $3a^2 \times -2a^n \times -4a^{2n-4} = 3 \times (-2) \times (-4) \times a^2 \times a^n \times a^{2n-4} = 24a^{3n-2}$.

Note that the product of more than two factors, called the *continued product*, is found in the same way as the product of two factors; also that, since every pair of negative factors produce a positive product, our product will be plus when we have an even number of negative factors and minus, when the number of negative factors is odd.

Rule for finding the product of any number of monomials. *Determine the sign of the product according to the Law of Signs in Multiplication; find the product of the numerical coefficients and to this product annex the letters, giving each letter the sum of the exponents of that letter in the various factors.*

EXERCISE 17

Find mentally, the product of:

1. $\begin{array}{r}2x\\ \underline{3x}\end{array}$	2. $\begin{array}{r}2x\\ \underline{-3x}\end{array}$	3. $\begin{array}{r}-2x\\ \underline{3x}\end{array}$	4. $\begin{array}{r}-2x\\ \underline{-3x}\end{array}$
5. $\begin{array}{r}3x^2\\ \underline{4x}\end{array}$	6. $\begin{array}{r}3x^2\\ \underline{-4x}\end{array}$	7. $\begin{array}{r}-3x^2\\ \underline{5x^2}\end{array}$	8. $\begin{array}{r}2ax^2\\ \underline{-3a^2x}\end{array}$
9. $\begin{array}{r}-4a^2x\\ \underline{5bx^2}\end{array}$	10. $\begin{array}{r}-3abc^2\\ \underline{-2a^2x}\end{array}$	11. $\begin{array}{r}3a^m\\ \underline{-2a^n}\end{array}$	12. $\begin{array}{r}5a^{m-2}\\ \underline{3a^2}\end{array}$
13. $\begin{array}{r}-2a^{2-m}\\ \underline{4a^{m+3}}\end{array}$	14. $\begin{array}{r}5a^2\\ \underline{-2a^{2-m}}\end{array}$	15. $\begin{array}{r}-3a\\ \underline{-4a^{3-m}}\end{array}$	16. $\begin{array}{r}x^{a+b}\\ \underline{x^{a-b}}\end{array}$

17. $2xy^2, -3x^2y^2$, and $4xyz$.

18. $-5cx, -3bx^2$ and $-2ax^3$.

19. $3abc, -2a^2c, 4b^2c$ and $-ac^2$.

20. $3xy, -\frac{2}{3}x^2, \frac{4}{7}y^2$ and $\frac{7}{8}xyz$.

21. $2x^n, -x^3, x^{2-n}, -4x^2$ and $-3x$.

22. $a^{x-1}, -3a^2, -4a^{2-x}$ and $2a^{3x}$.

23. $-2m^{1-a}, 5m^2, 3m^{2a}$, and $-5m$.

24. $\frac{1}{2}xy^n, \frac{2}{3}x^my^2, -\frac{3}{4}x^2y^{1-m}$ and $\frac{4}{5}x^{1-m}y$.

40. Degree of a Term.—The *degree* of a term is determined by the number of literal factors it contains.

Thus, $2a$ is of the *first degree*, $5ab$ is of the *second degree*, and $3ax^2y^3$ is of the *sixth degree*.

Degree of a Compound Expression.—The degree of a compound expression is the degree of the term having the largest number of literal factors.

Thus, $x^2 - 2x^3y + 4x^4y^2$, is an expression of the sixth degree.

Homogeneous Expression.—When all the terms of a compound expression are of the same degree, the expression is said to be *homogeneous*.

Thus, $x^4 - 2x^2y^2 + y^4$ is homogeneous because all the terms are of the fourth degree.

41. Dominant Letter.—When one letter in an expression is more important than any other, it is called the *dominant* letter. Thus, in, $x^2 + px + q$, x is the dominant letter and the expression is said to be of the *second degree in x*, as the highest power of the dominant letter is the second.

Arrangement of a Compound Expression.—An expression is *arranged* according to the powers of some letter, usually the dominant letter, when it is written so that the powers of that letter either *ascend* or *descend*.

Thus, $ax^3 - 2a^2x^2 + 2a^3x - a^4$, is said to be arranged according to the *descending powers of x;* the highest power of x being written first, then the next highest and so on. But $1 + x + x^2 + x^3$ is arranged according to the *ascending powers of x,* as the lowest power of x is written first.

MULTIPLICATION OF A POLYNOMIAL BY A MONOMIAL

42. Since $a(x + y) = ax + ay$
and $a(x - y) = ax - ay$.

We have the following

Rule.—*Multiply each term of the polynomial by the monomial multiplier and connect the various partial products by their proper signs.*

Thus, multiply $ax + bx^2 - cx^3$ by abc.

$$\begin{array}{l} ax + bx^2 - cx^3 \\ abc \\ \hline a^2bcx + ab^2cx^2 - abc^2x^3. \end{array}$$

In algebraic multiplications, it is usual to begin at the left and work towards the right.

The student will find it convenient, in some cases, essential, to arrange compound expressions according to the ascending or descending powers of the dominant letter.

EXERCISE 18

Multiply:

1. $a^2(ax + b)$.
2. $ac(ax - c)$.
3. $bc(a - b + c)$.
4. $abc(2a^2 - ab + b^2)$.
5. $-b^2(x^2 - ay + y^2)$.
6. $-3x^2(x^4 - x^2y^2 - y^4)$.
7. $4x^4y^2(a^2x^2 + ax - c)$.
8. $2x^5y^3z^2(x^2y + x^3y^2 - xy^3)$.
9. $5cx^3(10y^2 - 5xy - 10)$.
10. $-2a^3(a^4 - a^3b + a^2b^2)$.
11. $vt(s^2 - v^2s + at)$.
12. $uv(u + v - x^2 + y^2)$.
13. $-3z^2(a^2 - b^2 - c^2 + d^2)$.
14. $-4m^2n(m^2n^2 - mn^3 + n^4 + m^4)$.
15. $4x^2yz(-z^3 + 2xz - 5x^2yz^3)$.
16. $abd^4(3a^3bc^4 - 2b^4c^2x^3 + 5a^2b^3x - 3x^4y^2)$.
17. $-6mn^2(-4m^2n + 7a^2my - 6mn^2 - 2a^5m)$.
18. $5tu^4(3t^5u^6 + 2t^4u^7 - 4tu^3 + 2u^7)$.
19. $-7ak^3(6k^2 + 3ak^3 - 2a^2k^4 + 4a^2k^5)$.
20. $-a^5(a^2c^2 - ac^3 + c^4 - bc^5 - c^6)$.

MULTIPLICATION OF POLYNOMIALS BY POLYNOMIALS

43. If $a + b + c$ is to be multiplied by $a + b - c$, the multiplier $a + b - c$ may be represented by a single letter, as m, then, according to the preceding rule,

$$(a + b + c)\,(a + b - c) = (a + b + c)m = am + bm - cm$$

If the value of m is substituted,
$am + bm + cm = a(a + b - c) + b(a + b - c) + c(a + b - c) = a^2 + ab - ac + ab + b^2 - bc + ac + bc - c^2 = a^2 + 2ab + b^2 - c^2$.

Therefore, to **find the product of two polynomials,** we have the following

Rule.—*Multiply every term of the multiplicand by each term of the multiplier and add the partial products.*

Note, that for convenience in adding the partial products, like terms should be placed under each other.

Multiply $2x^2 + 3x - 1$ by $3x^2 - 2x + 3$.

$$\begin{array}{lllll}
2x^2 & +3x & -1 & & \\
\underline{3x^2} & \underline{-2x} & \underline{+3} & & \\
6x^4 & +9x^3 & -3x^2 & & \\
 & -4x^3 & -6x^2 & +2x & \\
 & & 6x^2 & +9x & -3 \\
\hline
6x^4 & +5x^3 & -3x^2 & +11x & -3
\end{array}$$

Multiply $2x^2$, the first term of the multiplicand by $3x^2$, the first term of the multiplier, giving $6x^4$. Then multiply $3x$, the second term of the multiplicand by $3x^2$, the first term of the multiplier, giving $9x^3$. Then multiply -1, the third term of the multiplicand by $3x^2$, the first term of the multiplier, giving $-3x^2$. This completes the first line of the partial products. Then multiply the three terms of the multiplicand by $-2x$, the second term of the multiplier, which gives the second line of partial products, like terms being written under each other. Similarly, multiply the three terms of the multiplicand by 3, the last term of the multiplier to obtain the third line of partial products. Adding these partial products gives the complete product, $6x^4 + 5x^3 - 3x^2 + 11x - 3$.

Note, that for convenience in adding the partial products, both multiplicand and multiplier should be arranged either according to the ascending or descending powers of the dominant letter.

EXERCISE 19

Multiply:

1. $(a + b)(a - b)$.
2. $(x + 3)(x + 4)$.
3. $(x + 2)(x - 5)$.
4. $(2x - 3)(3x + 2)$.
5. $(3x + 4)(3 - 2x)$.
6. $(a^2 + b^2)(a^2 - b^2)$.
7. $(x^2 + 2y)(y^2 + 2x)$.
8. $(2a^2 - b^2)(b^2 - 2a^2)$.
9. $(a^3 + b^3)(a + b)$.
10. $(a^4 - b^4)(a^2 - b^2)$.

11. $(x + y + z)(x + y - z)$.
12. $(x - 2y + 3z)(2x + y - z)$.
13. $(x^2 + xy + y^2)(x^2 - xy + y^2)$.
14. $(x^2 - xy + y^2)(x + y)$.
15. $(a^2 + ab + b^2)(a - b)$.
16. $(a^2 + b^2 + c^2)(a^2 - b^2 + c^2)$.
17. $(m^2 + 2m - 3)(m^2 - 3m + 2)$.
18. $(v^2 - uv + v^3)(u^2 + uv + u^3)$.
19. $(s^2 + rs + r^2 - r^3)(r^3 - r^2s + s^2)$.
20. $(x^2 + y^2 + 2xy + z)(x + y - z)$.
21. $(a^2 - 2abc + c^2 - b^2)(a - bc + c^2)$.
22. $(3x^2 - 2y^2 + 5z^2)(4x^2 - 2xy + y^2)$.
23. $(4a^2 + 10a^3b^3 - 5ab)(b^2 - 2a)$.
24. $(x^2 - 2xy + 6)(y^2 - 2xy + 4)$.
25. $(a^2 + b^2 - 1 + ab - a + b)(a + 1 + b)$.
26. $(2x^2 - 3z^2 + 5y^2)(4x^2 + 4z^2 - 3y^2)$.
27. $(a^2 + b^2 + c^2 + ab - ac)(a + b - c)$.
28. $(x^2 - xy + y^2 + x + y + 1)(x + y - 1)$.
29. $(m^3 - n^2 + n - 1)(n^3 - 3 + 2n^2)$.
30. $(4l - l^2m + 2l^3 - m^3)(m - 2l^2 + 3)$.

DIVISION OF ALGEBRAIC NUMBERS

44. In arithmetic, division is defined as the operation of finding how often one number is contained in another. While the same definition may be applied to algebraic numbers, it is usual to consider division as a reversal of multiplication, that is, finding one factor of the product, when the product and the second factor are known. The product becomes the dividend, the known factor, the divisor and the unknown factor, the quotient.

Since

$$\begin{aligned}
(+a) \times (+b) &= +ab \therefore +ab \div (+a) = +b \\
(-a) \times (-b) &= +ab \therefore +ab \div (-a) = -b \\
(+a) \times (-b) &= -ab \therefore -ab \div (+a) = -b \\
(-a) \times (+b) &= -ab \therefore -ab \div (-a) = +b
\end{aligned}$$

The **Law of Signs in Division** is therefore the same as in Multiplication, that is, *Like signs give plus, unlike signs give minus.*

By multiplication, the number of factors of the product is increased, hence we added the exponents of the same letters. In division, the number of factors is decreased, therefore

The Law of Exponents in Division is,

The exponent of a letter in the quotient is equal to the exponent of that letter in the dividend minus the exponent of that letter in the divisor.

Thus,

$$\frac{a^4}{a^2} = \frac{a \cdot a \cdot a \cdot a}{a \cdot a} = a \cdot a = a^{4-2} = a^2$$

If the exponents are letters, the subtraction must be indicated

As
$$\frac{a^m}{a^n} = a^{m-n}$$

DIVISION OF MONOMIALS

1. $15x^6 \div 5x^4$

$$\frac{15x^6}{5x^4} = 3x^{6-4} = 3x^2.$$

By following the above laws for signs and exponents, we have, like signs give plus, 15 divided by 5 gives us the coefficient 3 in the quotient and subtracting the exponent 4 of the divisor from the exponent 6 of the dividend, we get the exponent 2 of the quotient.

Similarly,

2. $-20x^3y^3 \div -5x^2y$

$$\frac{-20x^3y^3}{-5x^2y} = +4xy^2.$$

3. $-27a^5b^3 \div 9a^2b$

$$\frac{-27a^5b^3}{9a^2b} = -3a^3b^2.$$

4. $34x^{a-3} \div -17x^{a-5}$

$$\frac{34x^{a-3}}{-17x^{a-5}} = -2x^{(a-3)-(a-5)} = -2x^{a-3-a+5} = -2x^2.$$

5. $24a^2x^3y^m \div 8ax^3y^n$

$$\frac{24a^2x^3y^m}{8ax^3y^n} = 3ax^0y^{m-n} = 3ay^{m-n}.$$

The exponents of x being equal in dividend and divisor, the exponent of x in the quotient is 0. Since this is equivalent to dividing a quantity by its equal, it is evident that the quotient is 1; hence any quantity having a zero exponent is equal to 1 and may be omitted as a factor.

Rule for Finding the Quotient of Two Monomials:

Apply the Law of Signs; divide the numerical coefficient of the dividend by the numerical coefficient of the divisor; to the resulting quotient annex the letters, giving each letter an exponent equal to the difference between the exponents of that letter in dividend and in divisor.

EXERCISE 20

Perform the indicated divisions:—

1. $\dfrac{10a^2b^2}{5ab}$.
2. $\dfrac{12x^2y}{4xy}$.
3. $\dfrac{16m^2n^3}{-4m^2n}$.
4. $\dfrac{-25x^2yz^2}{-5xyz}$.
5. $\dfrac{32a^4b^2c^5}{8a^2bc^3}$.
6. $\dfrac{-44u^3v^3}{11uv^2}$.
7. $\dfrac{18x^3yz^3}{-6x^2yz^2}$.
8. $\dfrac{27r^2s^3t^2}{9r^2st^2}$.
9. $\dfrac{-24x^4y^5z^3}{-6x^2y^2z}$.
10. $\dfrac{39a^2bc^2}{13a^2c^2}$.
11. $\dfrac{14a^{n+2}}{7a^2}$.
12. $\dfrac{-11x^{a-2}}{x^{a-4}}$.
13. $\dfrac{21a^xb^2}{3a^yb}$.
14. $\dfrac{56x^{a-2}y^3}{7x^{2-a}y}$.
15. $\dfrac{18x^my^n}{6x^ny^m}$.
16. $\dfrac{26p^aq^b}{-8p^2q^2}$.
17. $\dfrac{54x^9y^7}{9x^my^n}$.
18. $\dfrac{-57x^3y^4}{19x^2y^6}$.

DIVISION OF A POLYNOMIAL BY A MONOMIAL

45. 1. Divide $3xy^4 - 6x^2y^5 + 12x^3y^6$ by $3y^3$.

Since the entire expression is to be divided by $3y^3$, each term of the dividend must be divided by the divisor.

$$\frac{3xy^4 - 6x^2y^5 + 12x^3y^6}{3y^3} = \frac{3xy^4}{3y^3} - \frac{6x^2y^5}{3y^3} + \frac{12x^3y^6}{3y^3} =$$

$xy - 2x^2y^2 + 4x^3y^3$.

2. Divide $10a^{3x+1} - 15a^{3x}$ by $5a^{2x}$.

Performing the division as in arithmetic,

$$\begin{array}{l} 5a^{2x})\underline{10a^{3x+1} - 15a^{3x}} \\ \qquad\;\; 2a^{x+1} - 3a^{x} \end{array}$$

Hence the **Rule for Dividing a Polynomial by a Monomial:** *Divide each term of the dividend by the divisor and connect the various quotients by their proper signs.*

EXERCISE 21

Divide:

1. $a^2 + ab$ by a.
2. $x^2 - 2xy$ by x.
3. $3m^3 - 6m^2n$ by $3m^2$.
4. $-4x^2y^2 - 10xy^3$ by $-2x^2y$.
5. $5a^2c^2 + 10a^3c - 15a^4$ by $5a^2$.
6. $12u^2v^2 - 15uv^3 - 18v^4$ by $-3v$.
7. $x^7y^2 + x^6y^3 - x^5y^4$ by x^3y.
8. $x^2 - xy + y^2z$ by $-x$.
9. $-9m^5n^2 + 12m^4n^3 - 15m^3n^4$ by $-3m^2n$.
10. $2x^n - 5x^{n-2}y$ by x^2.
11. $4a^2b^{x+4} + 6a^3b^{x+3} - 10a^4b^{x+2}$ by $2ab^2$.
12. $x^{m+n} - x^{m+n-1} + 2x^{m+n-2}$ by x^{n-1}.
13. $3a^{x+v+2} + 6a^{x+v+3} - 12a^{x+v+4}$ by $3a^{x+v+1}$.
14. $10x^{a+b}y^{a-b} - 15x^{a+2b}y^{a-2b} + 25x^{a+3b}y^{a-3b}$ by $-5x^{a+b}y^{a-b}$.

DIVISION OF A POLYNOMIAL BY A POLYNOMIAL

46. If we multiply $x + y + z$ by $m + n + p$, we have,

$$\begin{array}{lll}
x + y + z & & \\
\underline{m + n + p} & & \\
mx + my + mz & & \\
 & nx + ny + nz & \\
 & & \underline{px + py + pz} \\
\multicolumn{3}{l}{(mx + my + mz) + (nx + ny + nz) + (px + py + pz).}
\end{array}$$

Calling the first factor, $x + y + z$, the divisor, the product consisting of the sum of the three partial products, the dividend, then the other factor $(m + n + p)$ will be the quotient. If the dividend and divisor are known, we may find the quotient by noting what multiplier must be used to get the partial products. Thus we can see that $x + y + z$ must be multiplied by m, to produce the first partial product $mx + my + mz$. Therefore m is the first term of the quotient. Similarly, n is the multiplier needed to produce the second partial product $(nx + ny + nz)$ and p is the third needed multiplier.

Practically, we find the first term of our quotient by dividing the first term of the dividend by the first term of the divisor. Multiplying the divisor by this first term of the quotient gives us the first partial product. Subtract this from the total product and the other two partial products remain. Divide the first term of this remainder by the first term of the divisor and we get n, the second term of the quotient. Multiplying the divisor by this second term of the quotient, gives us the second partial product which when subtracted leaves the third partial product. Dividing the first term of this remainder by the first term of the divisor we obtain p, the third term of the quotient. The product of the divisor by this third term of the quotient gives us the third partial product, which being subtracted, leaves a remainder of zero.

The work, as actually done, is similar to an arithmetical long division.

1. Divide $a^2 + 7a + 12$ by $a + 3$

$$\begin{array}{r|l} a^2 + 7a + 12 & a + 3 \\ \underline{a^2 + 3a \phantom{{}+12}} & \underline{a + 4} \\ 4a + 12 & \\ \underline{4a + 12} & \end{array}$$

First arrange dividend and divisor in the same order, either according to the ascending or descending powers of the dominant letter. This is of great importance. Then divide the first term of the dividend, a^2, by the first term of the divisor a, giving a for the first term of the quotient. Multiply the divisor, $a + 3$, by a, the first term of the quotient, we have the first partial product, $a^2 + 3a$. Subtracting, we have the remainder, $4a + 12$. Divide $4a$, the first term of the remainder by a, the first term of the divisor, and we get 4, the second term of the quotient. Multiplying the divisor, $a + 3$, by this term of the quotient, we get $4a + 12$, the second partial product, which on subtraction leaves a remainder of zero. Therefore, the division comes out even and our quotient is $a + 4$.

Similarly,

2. Divide $x^2 - 2xy + y^2$ by $x - y$

$$\begin{array}{r|l} x^2 - 2xy + y^2 & x - y \\ \underline{x^2 - xy \phantom{{}+y^2}} & \underline{x - y} \\ -xy + y^2 & \\ \underline{-xy + y^2} & \end{array}$$

3. Divide $a^2 - b^2$ by $a - b$

$$\begin{array}{r|l} a^2 - b^2 \phantom{{}-b^2} & a - b \\ \underline{a^2 - ab \phantom{{}-b^2}} & \underline{a + b} \\ ab - b^2 & \\ \underline{ab - b^2} & \end{array}$$

Note that the multiplication of the divisor, $a - b$, by a, the first term of the quotient, produces a term, $-ab$, whose like does not occur in the dividend. This term must be written before $-b^2$ in the remainder as it contains a higher power of a, our expressions having been arranged according to the descending powers of a.

The same principle must be observed in the following problem:

4. Divide $a^3 + b^3 + c^3 - 3abc$ by $a + b + c$

$$
\begin{array}{l|l}
a^3 - 3abc + b^3 + c^3 & a + b + c \\ \cline{2-2}
\underline{a^3 + a^2b + a^2c} & \underline{a^2 - ab - ac + b^2 - bc + c^2} \\
-a^2b - a^2c - 3abc + b^3 + c^3 & \\
\underline{-a^2b - ab^2 - abc} & \\
-a^2c + ab^2 - 2abc + b^3 + c^3 & \\
\underline{-a^2c - abc - ac^2} & \\
\qquad ab^2 - abc + ac^2 + b^3 + c^3 & \\
\qquad \underline{ab^2 + b^3 + b^2c} & \\
\qquad\quad -abc + ac^2 - b^2c + c^3 & \\
\qquad\quad \underline{-abc - b^2c - bc^2} & \\
\qquad\qquad ac^2 + bc^2 + c^3 & \\
\qquad\qquad \underline{ac^2 + bc^2 + c^3} &
\end{array}
$$

5. Divide $a^{m+n}b^n - 4a^{m+n-1}b^{2n} - 27a^{m+n-2}b^{3n} + 42a^{m+n-3}b^{4n}$ by $a^m + 3a^{m-1}b^n - 6a^{m-2}b^{2n}$

$$
\begin{array}{l|l}
a^{m+n}b^n - 4a^{m+n-1}b^{2n} - 27a^{m+n-2}b^{3n} + 42a^{m+n-3}b^{4n} & a^m + 3a^{m-1}b^n - 6a^{m-2}b^{2n} \\ \cline{2-2}
\underline{a^{m+n}b^n + 3a^{m+n-1}b^{2n} - 6a^{m+n-2}b^{3n}} & \underline{a^nb^n - 7a^{n-3}b^{4n}} \\
\qquad -7a^{m+n-1}b^{2n} - 21a^{m+n-2}b^{3n} + 42a^{m+n-3}b^{4n} & \\
\qquad \underline{-7a^{m+n-1}b^{2n} - 21a^{m+n-2}b^{3n} + 42a^{m+n-3}b^{4n}}. &
\end{array}
$$

EXERCISE 22

Divide:

1. $x^2 + 5x + 6$ by $x + 2$.
2. $x^2 - x - 6$ by $x - 3$.
3. $a^2 + 2ab + b^2$ by $a + b$.
4. $a^2 - 2ab + b^2$ by $a - b$.
5. $x^2 - y^2$ by $x + y$.
6. $9a^2 + 12a + 4$ by $3a + 2$.
7. $4a^2 - 12a + 9$ by $2a - 3$.
8. $3a^2 + 7a + 2$ by $a + 2$.
9. $x^2 - 4y^2$ by $x - 2y$.
10. $x^3 - y^3$ by $x - y$.
11. $3x^2 - 4x - 4$ by $2 - x$.
12. $-6x^2 + x + 12$ by $3x + 4$.

13. $a^4 - b^4$ by $a^2 - b^2$.

14. $2x^3 + x^2y^2 + 4xy + 2y^3$ by $y^2 + 2x$.

15. $a^4 + a^3b + b^4 + ab^3$ by $a^3 + b^3$.

16. $x^2 + 2xy + y^2 - z^2$ by $x + y + z$.

17. $2x^2 - 2y^2 - 3z^2 - 3xy + 5xz + 5yz$ by $2x + y - z$.

18. $x^4 + y^4 + x^2y^2$ by $x^2 + xy + y^2$.

19. $a^4 - b^4 + c^4 + 2a^2b^2$ by $a^2 - b^2 + c^2$.

20. $m^4 - m^3 + 5m^2 - 5m - 6$ by $m^2 + 2m - 3$.

21. $-u^3v + u^2v^3 + uv^3 + uv^4 + u^3v^2 - u^4v + u^3v^3$ by $v^2 - uv + v^3$.

22. $r^5 - r^6 - r^2s^3 - r^5s + s^4 + rs^3 + r^2s^2 - r^3s^2$ by $s^2 + rs + r^2 - r^3$.

23. $x^3 + 3xy^2 + 3x^2y + xz + y^3 + yz - x^2z - y^2z - 2xyz - z^2$ by $x^2 + y^2 + 2xy + z$.

24. $a^3 - 3a^2bc + ac^2 - ab^2 + 2ab^2c - bc^3 + b^3c + a^2c^2 - 2abc^3 + c^4 - b^2c^2$ by $a^2 - 2abc + c^2 - b^2$.

25. $x^3 + 3xy + y^3 - 1$ by $x^2 - xy + y^2 + x + y + 1$.

26. $x^6 - y^6$ by $x - y$.

27. $6a^{2n+1} - 13a^{2n} + 6a^{2n-1}$ by $3a^{n+1} - 2a^n$.

28. $12a^{3n} + 13a^{2n} + 3a^n$ by $3a^n + 1$.

29. $6x^{n+1} - 5x^n - 6x^{n-3} + 13x^{n-2} - 6x^{n-1}$ by $2 + 2x^2 - 3x$.

30. $x^{n+2} - 3x^{n+1} - x^n + 2x^{n-1} - 4x^{n-2}$ by $x^2 - x + 1$.

REVIEW EXERCISE V

1. Multiply $m^3 + 2m^2n + 2mn^2 + n^3$ by $m^3 - 2m^2n + 2mn^2 - n^3$.

2. Divide $y^4 - 5y^2z^2 + 4z^4$ by $y^2 - yz - 2z^2$.

3. Multiply $4a^7y - 32ay^4 - 8a^5y^2 + 16a^3y^5$ by $a^6y^2 + 4a^2y^4 + 4a^4y^3$.

4. Divide $x^5 - 41x - 120$ by $x^2 + 4x + 5$.

5. Multiply $a^{2m} - a^mb^m + b^{2m}$ by $a^m + b^m$.

6. Multiply $4a^{m+1}b^2 + a^{m-2}b - 2a^{m+4}b^3$ by $3a^mb^2 - a^{m+3}b^3 - 5a^{m+6}b^2$.

7. Simplify $(m + n)m - \{(m - n)^2 - (n - m)n\}$.

8. Divide $x^4 - y^2 - 9x^2 - 6xy$ by $x^2 + 3x + y$.

9. Multiply $ax^{2n} + ay^{2n}$ by $ax^{2n} - ay^{2n}$.

10. Divide $x^4 + 81$ by $x - 3$.

11. Multiply $(\frac{3}{4}x^2 - xy + \frac{4}{3}y^2)(\frac{4}{3}x + \frac{3}{4}y)$ and test result with values $x = 6$, and $y = 6$.

12. Multiply $3x^2 + x^3 - 2x - 4$ by $2x + 1 + 4x^3 + 3x^2$.

13. Divide $2a^4 + 6a^2b^2 - 5a^3b + b^4 - 4ab^3$ by $a^2 + b^2 - ab$.

14. Divide $4x^5 - 5x^4 + 45x^2 - 10 + x + 20x^3$ by $3x - 2 + 4x^2$.

15. Divide $x^5 + 50 - 70x + 37x^2$ by $10 - 2x + x^2$.

16. Divide $5a - 10a^3 - 4a^5 + 13a^2 + 4a^4 + a^6$ by $1 + 3a - a^2$.

17. Multiply $a - a^4x^4 + x^8 + 1$ by $a^4 + x^4 - 1$.

18. Divide $2x^3 - 10 - 6x^2 + x^4 + 11x$ by $2 + x^2 - x$.

19. Multiply $0.1x^2 + 0.04x + 0.5$ by $0.1x^2 + 4x + 0.05$.

20. Divide $x^{4n} - x^{3n}y^m + x^ny^{3m} - y^{4m}$ by $x^n - y^m$.

21. Multiply $\frac{2}{3}x^2 + xy + \frac{3}{2}y^2$ by $\frac{1}{3}x - \frac{1}{2}y$.

CHAPTER VI

SPECIAL RULES FOR MULTIPLICATION AND DIVISION

47. If we multiply $a + b$ by $a + b$, that is, square $a + b$,

$$\begin{array}{l} a + b \\ \underline{a + b} \\ a^2 + ab \\ \underline{\quad + ab + b^2} \\ a^2 + 2ab + b^2 \end{array}$$

the product is seen to consist of three terms, the first term being the square of a, the first term of the binomial $a + b$, the second term of the product, $2ab$, is twice the product of the two terms of the binomial $a + b$, and the third term of the product, b^2, is the square of the second term of the binomial $a + b$.

It is evident, that if, instead of $a + b$, the binomial had been $x + y$, the *form* of the product would have been the same, the letters changing to x and y instead of a and b. Experience then, teaches us that the square of the sum of two numbers is always equal to the square of the first number plus twice the product of the first by the second number, plus the square of the second number. Hence it is no longer necessary to multiply a binomial of this form by itself to find its square, but simply to write the product according to the rule found above. In the same way, we can find rules for writing the products of many other forms of multiplications or the quotients of certain forms of division. This so-called **Inspection Method** of writing the product or quotient according to rule, is of great importance in Algebra, as most of our work is done in this way. The student should take great pains to acquire the ability of recognizing the forms of algebraic expressions and memorize the rules that apply to them.

48. The Square of the Sum of Two Numbers.

Since, as shown above,

$$(a + b)^2 = a^2 + 2ab + b^2$$

we have the following

Rule 1. *The square of the sum of two numbers is equal to the sum of the squares of the numbers plus twice their product.*

EXERCISE 23

Write, by inspection,

1. $(x + y)^2$.
2. $(a + 2b)^2$.
3. $(3a + 4c)^2$.
4. $(a^2 + b^2)^2$.
5. $(2x + xy)^2$.
6. $(3a^2 + 2ab)^2$.
7. $(ax^2 + 2ay^2)^2$.
8. $(a^3 + 2b^3)^2$.
9. $(5a^2b^2 + 4b^2c^2)^2$.
10. $(3x^2yz + 2xy^2z)^2$.

49. The Square of the Difference of Two Numbers.

If we multiply $a - b$ by $a - b$,

$$\begin{array}{l}
a \quad - b \\
\underline{a \quad - b \qquad\;\;} \\
a^2 - ab \\
\underline{\quad\;\; - ab + b^2} \\
a^2 - 2ab + b^2
\end{array}$$

the product differs from the square of $a + b$, in the sign of the middle term, $-2ab$.

Hence, the following

Rule 2.—*The square of the difference of two numbers is equal to the sum of the squares of the numbers minus twice their product.*

EXERCISE 24

Write, by inspection,

1. $(x - y)^2$.
2. $(2a - b)^2$.
3. $(3a - 4b)^2$.
4. $(a^2 - b^2)^2$.
5. $(2x - xy)^2$.
6. $(2x^2 - 3xy)^2$.
7. $(ab^2 - 2ac^2)^2$.
8. $(x^3 - y^3)^2$.
9. $(4a^2b^2 - 5b^2c^2)^2$.
10. $(7r^2st - 5rs^2t^2)^2$.

50. The Product of the Sum and Difference of Two Numbers.

Multiply $a + b$ by $a - b$.

$$\begin{array}{rrr} a & +\,b & \\ a & -\,b & \\ \hline a^2 & +\,ab & \\ & -\,ab & -\,b^2 \\ \hline a^2 & & -\,b^2 \end{array}$$

Here the middle term disappears and our product consists of the square of the first number minus the square of the second number, hence,

Rule 3.—*The product of the sum and difference of two numbers is equal to the difference of their squares.*

EXERCISE 25

Write, by inspection, the product of:

1. $(x + y)(x - y)$.
2. $(2a + b)(2a - b)$
3. $(3x + 4y)(3x - 4y)$.
4. $(a^2 + b^2)(a^2 - b^2)$.
5. $(2x^2 - y^2)(2x^2 + y^2)$.
6. $(3xy - 4yz)(3xy + 4yz)$.
7. $(x^3 + y^3)(x^3 - y^3)$.
8. $(6a^2b^2 + 5b^2c^2)(6a^2b^2 - 5b^2c^2)$.
9. $(1 + a^2)(1 - a^2)$.
10. $(5u^2v^2 + 1)(5u^2v^2 - 1)$.

51. The Product of Two Binomials whose First Terms are Alike.

Multiply $a + 3$ by $a - 1$.

$$\begin{array}{rrr} a & +\,3 & \\ a & -\,1 & \\ \hline a^2 & +\,3a & \\ & -\,a & -\,3 \\ \hline a^2 & +\,2a & -\,3 \end{array}$$

Examining this product, we get the following

Rule 4.—*The product of two binomials, whose first terms are alike, consists of three terms; the first is the square of the first term of the binomials, the second term of the product* *is the first term of the binomials with a coefficient equal to the algebraic sum of the two second terms of the binomials, and the third term of the product is the product of the two second terms of the binomials.*

Multiply $x + a$ by $x - b$.

As the second terms of these binomials cannot be combined by addition, like arithmetical numbers, the algebraic addition must be indicated.

Thus, $(x + a)(x - b) = x^2 + (a - b)x - ab$

Similarly, the rule must apply even if the terms of the binomials are compound expressions, consisting of more than one term.

Multiply $[x + 3(y + z)][x - 5(y + z)]$

According to rule,

$$[x + 3(y + z)][x - 5(y + z)] = x^2 - 2(y + z)x - 15(y + z)^2$$

EXERCISE 26

Write, by inspection, the product of:

1. $(x + 5)(x + 3)$.
2. $(a - 5)(a + 2)$.
3. $(a + 8)(a - 3)$.
4. $(x - 2)(x - 5)$.
5. $(x + a)(x + 3a)$.
6. $(x - 2y)(x - 3y)$.
7. $(x + 4a)(x - 3a)$.
8. $(a + 2x)(a - 5x)$.
9. $(2x + y)(2x + 3y)$.
10. $(2x - 3y)(2x - 5y)$.
11. $(3a + 5b)(3a - 2b)$.
12. $(5x + 2y)(5x - 6y)$.
13. $(1 + ab)(1 + 2ab)$.
14. $(x^2 + 2a)(x^2 - 3b)$.
15. $(2x^2 - 3xy)(2x^2 + 5xy)$.
16. $(a + c)(a + b)$.
17. $(a - b)(a - c)$.
18. $[(a + b) + 8][(a + b) - 5]$.
19. $[x + 5(y + z)][x - 7(y + z)]$.
20. $[(a + b) - 4(c + d)][(a + b) - 3(c + d)]$.

52. The Product of Any Two Binomials.

Multiply $2a + 3b$ by $3a - 4b$.

In this case the product is not written according to any special rule. The multiplication is performed in the usual way, except, that the numbers are not written under each other as in an ordinary multiplication and that the partial products, if like terms, are added mentally.

Thus $(\overline{2a + \underline{3b)(3a} - 4b}) = 6a^2 + ab - 12b^2$.

Here, the partial products, as indicated by lines, $9ab$ and $-8ab$, being like terms, are added mentally, giving ab as the middle term. If the partial products are unlike terms, they are written as they are produced by multiplication.

Thus $(a + 2b)(2a - c) = 2a^2 + 4ab - ac - 2bc$.

EXERCISE 27

Write the product of:

1. $(2x + y)(x + 2y)$.
2. $(3a + b)(2a - 3b)$.
3. $(5x + 3y)(3x - 5y)$.
4. $(2x - 5y)(3x - 4y)$.
5. $(2a^2 + 3b^2)(3a^2 + 4b^2)$.
6. $(5x^2 - 2y^2)(3x^2 - 5y^2)$.
7. $(1 - x^2y^2)(3 + 2x^2y^2)$.
8. $(a^2 - b^2)(a - b)$.
9. $(a^3 + b^3)(a^2 + b^2)$.
10. $(2x + y)(3x - z)$.
11. $(8c + 3d)(5c - 8d)$.
12. $(2ac + 5bd)(3ac - 4bd)$.
13. $(5x - 7y)(5x - 8y)$.
14. $(a^2c^2 + 5d)(2a^2c^2 - 8d)$.
15. $[(a + b) + 5][2(a + b) - 8]$.

53. Square of Any Polynomial.—If any polynomial, such as $x + y - z$, is multiplied by itself, the product is, $x^2 + 2xy - 2xz + y^2 - 2yz + z^2$.

This product consists of the sum of the squares of the several terms and the algebraic sum of the double products of each term by each of the others.

Hence, to square any polynomial, we have the following

Rule 5.—*The square of a polynomial is equal to the sum of the squares of the several terms, plus twice the product of each term by each of the following terms.*

Thus, $(2a - 3b + c - 4d + 5e)^2 = (2a)^2 + (-3b)^2 + c^2 + (-4d)^2 + (5e)^2 + 2(2a)(-3b + c - 4d + 5e) + 2(-3b)(c - 4d + 5e) + 2c(-4d + 5e) + 2(-4d)(5e) = 4a^2 + 9b^2 + c^2 + 16d^2 + 25e^2 - 12ab + 4ac - 16ad + 20ae - 6bc + 24bd - 30be - 8cd + 10ce - 40de$.

EXERCISE 28

Write the square of:

1. $a + b + c$.
2. $x - y + z$.
3. $a + b - c$.
4. $a - b + 3$.
5. $x + 2y - 4$.
6. $2a + 3b - c$.
7. $3ab - c + 4d$.
8. $a^2 + b^2 + c^2$.
9. $2a - b - c$.
10. $x - 2y - 3z$.
11. $a^2 + b^2 - c^2$.
12. $x^2 - y^2 - z^2$.
13. $x^2 - xy + y^2$.
14. $a + b - c + d$.
15. $a - b + c - 1$.
16. $x^2 + xy - y^2 + 1$.
17. $2x - 3y + 4z - 5$.
18. $a^3 + b^3 + c^3 - 2abc$.
19. $x + y - 2xy + y^2$.
20. $a - b + 3c + 2e - f$.

54. The product of two polynomials which may be arranged in the form of the sum and difference of two numbers.

If two polynomials can be written as the sum and difference of two numbers, their product may be written according to *Rule* 3.

Thus, $(x + y + z)(x - y - z)$, may be written $[x + (y + z)]$ $[x - (y + z)]$, where x is considered one number and $y + z$ the other number. The first factor is the sum of these two numbers and the second factor is the difference of these two numbers. The rule tells us that the product of these two factors is the difference of the squares of the two numbers, hence we write the square of the first number minus the square of the second number,

$$x^2 - (y + z)^2$$

Squaring the second number, according to *Rule* 1,

$$x^2 - (y^2 + 2yz + z^2) = x^2 - y^2 - 2yz - z^2, \text{ the product.}$$

Similarly,

$$(a - 2b + 3c - d)(a + 2b + 3c + d) =$$
$$[(a + 3c) - (2b + d)][(a + 3c) + (2b + d)] =$$
$$(a + 3c)^2 - (2b + d)^2 = (a^2 + 6ac + 9c^2) - (4b^2 + 4bd + d^2) =$$
$$a^2 + 6ac + 9c^2 - 4b^2 - 4bd - d^2$$

EXERCISE 29

Write the product of:

1. $(a + b + c)(a + b - c)$.
2. $(a - b + c)(a - b - c)$.

3. $(a + b + c)(a - b - c)$.
4. $(a + b - c)(a - b + c)$.
5. $(x + y + z)(x - y + z)$.
6. $(x + y - z)(x - y - z)$.
7. $(ab + c + 1)(ab + c - 1)$.
8. $(1 + a + b)(1 - a + b)$.
9. $(1 + a - b)(1 - a - b)$.
10. $(x^2 + xy + y^2)(x^2 - xy + y^2)$.
11. $(m^2 + m + 2)(m^2 - m - 2)$.
12. $(3a - 2b + c)(3a + 2b - c)$.
13. $(m + n + p + q)(m + n - p - q)$.
14. $(m - n + p - q)(m - n - p + q)$.
15. $(m + n + p + q)(m - n - p + q)$.
16. $(m + n + p - q)(m - n - p - q)$.
17. $(2a - 3b - 5c - d)(2a - 3b + 5c + d)$.
18. $(2a + 3b - 5c - d)(2a - 3b - 5c + d)$.

SPECIAL RULES FOR DIVISION

55. If a number, which is the product of two factors, is used as a dividend and one of the factors is used as a divisor, the other factor will be the quotient. Thus, since $15 = 3 \times 5$, $15 \div 3 = 5$ or $15 \div 5 = 3$.

Therefore, if any algebraic expression is recognized as a product, written according to any of the foregoing rules, that expression, divided by one of its factors, will yield the other factor as a quotient.

Since $(a + b)(a - b) = a^2 - b^2$.
$(a^2 - b^2) \div (a + b) = (a - b)$ or
$(a^2 - b^2) \div (a - b) = (a + b)$

Hence the following rule for the **Difference of Two Squares.**

Rule 1.—*The difference of the squares of two numbers is divisible by the sum of the numbers and the quotient is the difference of the numbers.*

Rule 2.—*The difference of the squares of two numbers is divisible by the difference of the numbers and the quotient is the sum of the numbers.*

Thus, divide $(25u^4v^4 - 1)$ by $(5u^2v^2 + 1)$.

Since $25u^4v^4$ is the square of $5u^2v^2$ and 1 is the square of 1, $25u^4v^4 - 1$ is the difference of two squares, $5u^2v^2 + 1$ is the sum of the two numbers and the quotient is the difference of the two numbers,

$$\frac{25u^4v^4 - 1}{5u^2v^2 + 1} = 5u^2v^2 - 1$$

Similarly

$$\frac{(m+n)^2 - (p+q)^2}{(m+n) + (p+q)} = (m+n) - (p+q) = m + n - p - q$$

and

$$\frac{(2a-3b)^2 - (5c-d)^2}{(2a-3b) - (5c-d)} = (2a - 3b) + (5c - d) =$$
$$2a - 3b + 5c - d.$$

EXERCISE 30

Write, by inspection, the quotient of:

1. $\dfrac{4a^2 - b^2}{2a + b}$.
2. $\dfrac{9x^2 - 16y^2}{3x - 4y}$.
3. $\dfrac{a^4 - b^4}{a^2 - b^2}$.
4. $\dfrac{4x^4 - y^4}{2x^2 - y^2}$.
5. $\dfrac{9x^2y^2 - 16y^2z^2}{3xy + 4yz}$.
6. $\dfrac{x^6 - y^4}{x^3 - y^2}$.
7. $\dfrac{1 - a^4}{1 + a^2}$.
8. $\dfrac{36a^6b^2 - 1}{6a^3b - 1}$.
9. $\dfrac{a^2 - (b+c)^2}{a + (b+c)}$.
10. $\dfrac{(a-b)^2 - c^2}{(a-b) - c}$.
11. $\dfrac{1 - (x-y)^2}{1 - (x-y)}$.
12. $\dfrac{(2a+3b)^2 - (3c-2d)^2}{(2a+3b) + (3c-2d)}$.
13. $\dfrac{(a+b)^4 - (c-d)^6}{(a+b)^2 + (c-d)^3}$.

56. By performing the actual division,

$$\begin{array}{l|l} a^3 + b^3 & a + b \\ \underline{a^3 + a^2b} & \overline{a^2 - ab + b^2} \\ \quad -a^2b + b^3 & \\ \quad \underline{-a^2b - ab^2} & \\ \qquad\quad ab^2 + b^3 & \\ \qquad\quad \underline{ab^2 + b^3} & \end{array} \qquad \begin{array}{l|l} a^3 - b^3 & a - b \\ \underline{a^3 - a^2b} & \overline{a^2 + ab + b^2} \\ \quad a^2b - b^3 & \\ \quad \underline{a^2b - ab^2} & \\ \qquad\quad ab^2 - b^3 & \\ \qquad\quad \underline{ab^2 - b^3} & \end{array}$$

we see that the sum of two cubes is divisible by the sum of the numbers and the difference of two cubes is divisible by the difference of the two numbers and that the quotients have the form given in the following

Rule 3.—The sum of the cubes of two numbers *is divisible by the sum of the numbers and the quotient is the sum of the squares of the numbers minus their product.*

Rule 4.—The difference of the cubes of two numbers *is divisible by the difference of the numbers and the quotient is the sum of the squares of the numbers plus their product.*

Note the resemblance of these quotients to the square of a binomial like $a + b$. $(a + b)^2 = a^2 + 2ab + b^2$ but $\frac{a^3 - b^3}{a - b} = a^2 + ab + b^2$, the middle terms only differing in their coefficients.

EXERCISE 31

Write, by inspection, the quotient of:

1. $\frac{x^3 + 8y^3}{x + 2y}$.
2. $\frac{8a^3 - b^3}{2a - b}$.
3. $\frac{1 - 27x^3}{1 - 3x}$.
4. $\frac{8c^3 + 1}{2c + 1}$.
5. $\frac{8x^3 - 27y^3}{2x - 3y}$.
6. $\frac{64a^3 + 8b^3}{4a + 2b}$.
7. $\frac{x^3y^3 - z^3}{xy - z}$.
8. $\frac{27a^3b^3 + 8c^3}{3ab + 2c}$.
9. $\frac{a^6 - b^3}{a^2 - b}$.
10. $\frac{x^9 + 8y^6}{x^3 + 2y^2}$.
11. $\frac{(a + b)^3 + (c - d)^3}{(a + b) + (c - d)}$.
12. $\frac{1 - (x - y)^6}{1 - (x - y)^2}$.

THE SUM AND DIFFERENCE OF TWO LIKE POWERS

57. The sum and difference of two like powers may have any one of four possible forms, that is

1. The sum of two even powers, as $a^4 + b^4$
2. The sum of two odd powers, as $a^5 + b^5$.
3. The difference of two odd powers, as $a^5 - b^5$.
4. The difference of two even powers, as $a^4 - b^4$.

By actual trial, we find that each form, except the first, is divisible, either by the sum of the two numbers, $a + b$, or the difference of the numbers, $a - b$,

$a^5 + b^5$ is divisible by $a + b$,
$a^5 - b^5$ is divisible by $a - b$, and
$a^4 - b^4$ is divisible by both $a + b$ and $a - b$.

Thus,

$$\frac{a^5 + b^5}{a + b} = a^4 - a^3b + a^2b^2 - ab^3 + b^4$$

$$\frac{a^5 - b^5}{a - b} = a^4 + a^3b + a^2b^2 + ab^3 + b^4$$

$$\frac{a^4 - b^4}{a + b} = a^3 - a^2b + ab^2 - b^3$$

$$\frac{a^4 - b^4}{a - b} = a^3 + a^2b + ab^2 + b^3$$

These quotients are seen to follow a definite

Rule 5.—*The sum of two like odd powers is divisible by the sum of the two numbers.*

The difference of two like odd powers is divisible by the difference of the two numbers.

The difference of two like even powers is divisible by the sum and the difference of the two numbers.

If the divisor is the sum of the two numbers, the terms of the quotient are alternately plus and minus.

If the divisor is the difference of the two numbers, the terms of the quotient are all plus.

The first term of each quotient is obtained by dividing the first term of the dividend by the first term of the divisor, and in the succeeding terms of the quotient, the powers of the first number decrease by 1 *and the powers of the second number increase by* 1.

Thus, $\frac{(x^7 + y^7)}{x + y} = x^6 - x^5y + x^4y^2 - x^3y^3 + x^2y^4 - xy^5 + y^6$

Since the divisor is the sum of the numbers x and y, the terms of the quotient are alternately plus and minus. The first term of the dividend, x^7, divided by the first term of the divisor, x, gives x^6 as the first term of the quotient. We may assume that x^6 contains the zero power of y. In the second term of the quotient, the power of x decreases by 1 and the power of the second number y, increases by 1, hence x^5y, and so on to the last term, which contains the zero power of x and the 6th power of y.

Similarly, $\dfrac{16 - x^4}{2 - x} = 2^3 + 2^2x + 2x^2 + x^3 = 8 + 4x + 2x^2 + x^3$

and
$$\begin{aligned}\frac{a^6 - 1}{a + 1} &= a^5 - a^4 \cdot 1 + a^3 \cdot 1^2 - a^2 \cdot 1^3 + a \cdot 1^4 - 1^5\\ &= a^5 - a^4 + a^3 - a^2 + a - 1\end{aligned}$$

$a^6 + b^6$, the sum of two even powers, is not divisible by $a + b$ or $a - b$, but if we consider it as $(a^2)^3 + (b^2)^3$, the sum of two like odd powers, it is divisible by the sum of the two numbers, $a^2 + b^2$.

$$\frac{a^6 + b^6}{a^2 + b^2} = \frac{(a^2)^3 + (b^2)^3}{a^2 + b^2} = (a^2)^2 - (a^2)(b^2) + (b^2)^2 = a^4 - a^2b^2 + b^4$$

EXERCISE 32

Write, by inspection, the quotient of:

1. $\dfrac{x^5 + y^5}{x + y}$.
2. $\dfrac{x^4 - y^4}{x + y}$.
3. $\dfrac{x^6 - y^6}{x - y}$.
4. $\dfrac{1 - x^7}{1 - x}$.
5. $\dfrac{a^5 - 1}{a - 1}$.
6. $\dfrac{32x^5 + y^5}{2x + y}$.
7. $\dfrac{16a^4 - 81b^4}{2a - 3b}$.
8. $\dfrac{x^8 - y^4}{x^2 + y}$.
9. $\dfrac{a^{12} + b^{12}}{a^4 + b^4}$.
10. $\dfrac{16x^8 - 1}{2x^2 + 1}$.
11. $\dfrac{1 - c^6}{1 - c}$.
12. $\dfrac{1 + x^5}{1 + x}$.

REVIEW EXERCISE VI

By inspection:

1. Expand $(x^3 - 2x^2 + 3x + 4)^2$.
2. Multiply $(a + b - 2)$ by $(a - b - 2)$.
3. Find the product $(a - b^2 + c^2 - d)(a - b^2 - c^2 + d)$.
4. Find the quotient of $(a^6 - b^6) \div (a - b)$.
5. Divide $x^5 + 32$ by $x + 2$.
6. Find value of $(y + c + d)(y - c - d)$.
7. Divide $x^6 + y^6$ by $x^2 + y^2$.
8. Expand $(ax + bx - cx)^2$.
9. Find the product $(c - d + 3)(c + d + 3)$.
10. Find the product $(x^2 + xy + y^2)(x^2 - xy + y^2)$.

CHAPTER VII

FACTORING

58. The **Factors** of any expression are the parts which, when multiplied together, will produce the expression.

It is important to note the difference between the terms of an algebraic expression and the factors of an algebraic expression. The terms of an expression are algebraically added to produce the expression, while its factors must be multiplied to produce it.

An expression is **rational** when none of its terms contains an indicated root, such as a square root or cube root or any other root.

Factors of rational and integral expressions only are considered in this chapter.

From the definition of the factors of an algebraic expression, it is evident that any factor of an expression is an exact divisor of that expression.

Factoring an expression usually means to break it up into its **prime factors, prime** having the same meaning here as in Arithmetic.

In the case of a monomial, the prime factors may be found by inspection. Thus, the factors of $10xy^2$ are, 5, 2, x, y and y.

Most polynomials may also be factored by inspection, usually by recognizing the form as being produced by some one of the special rules of multiplication.

As a knowledge of factoring is exceedingly important, the student should carefully note the characteristics of the various forms considered in this chapter.

59. Case 1. A Polynomial whose terms contain a Common Monomial Factor.—*Common* here means *belonging to each*, that is a common factor is one found in every term of the expression.

Resolve into factors: $4a^2b^2 - 6ab$. By inspection, 2, a, and b, are seen to be factors of each term, therefore $2ab$ is said to be the *common monomial factor* of $4a^2b^2 - 6ab$. Since each term contains $2ab$, the expression may be divided by $2ab$ and the quotient obtained will be the second factor.

$$4a^2b^2 - 6ab = 2ab(2ab - 3).$$

Rule.—*When a polynomial contains a common monomial factor, determine this factor by inspection. Obtain the second factor by dividing each term of the expression by the common monomial factor.*

EXERCISE 33

Resolve into factors:

1. $2x^3 + 4x^2$.
2. $3a^2 - 6a^3$.
3. $a^3x + a^2b$.
4. $a^2cx - ac^2$.
5. $abc - b^2c + bc^2$.
6. $2a^3bc - a^2b^2c + ab^3c$.
7. $b^2x^2 - ab^2y + b^2y^2$.
8. $3x^6 - 3x^4y^2 - 3x^2y^4$.
9. $4a^2x^5y^2 + 4ax^4y^2 - 4cx^3y^2$.
10. $2x^7y^4z^2 + 2x^8y^5z^2 - 2x^6y^6z^2$.
11. $50cx^3y^2 - 25cx^4y - 50cx^3$.
12. $2a^7 - 2a^6b + 2a^5b^2$.
13. $s^2tv - stv^3 + at^2v$.
14. $u^2v + uv^2 - uvx^2 + uvy^2$.
15. $3a^2z^2 - 3b^2z - 3c^2z^2 + 3d^2z^2$.
16. $4m^4n^3 - 4m^3n^4 + 8m^2n^5 + 12m^6n$.
17. $4x^2yz^3 + 8x^3yz^2 - 20x^4y^2z^4$.
18. $3a^4b^5cd^4 - 2ab^5c^2d^4x^3 + 5a^3b^4cd^4$.
19. $24m^3n^3 + 42a^2m^2n^2y - 36m^2n^4$.
20. $15t^6u^{10} + 10t^5u^{11} - 20t^2u^7 + 10tu^{11}$.
21. $42ak^5 + 21a^2k^6 - 14a^3k^7 + 28a^3k^8$.
22. $a^7c^2 - a^6c^3 + a^5c^4 - a^5bc^5 - a^5c^6$.

60. Case 2. A Trinomial which is a Perfect Square.—Referring to the rule for squaring a binomial, we see that the square of any binomial consists of the square of the first term, plus or minus twice the product of the two terms plus the square of the second term. Hence any trinomial [illegible] and third terms are [illegible] product of the square

[illegible] perfect square, since $16x^2$ [illegible] middle term, $24xy$, is twice the

58. [illegible] $9y^2$ is the product of two equal [illegible] whe[illegible] $3y$ or $16x^2 - 24xy + 9y^2 = (4x - 3y)^2$.

[illegible] factor such a perfect square, we have the

[illegible] To find one of the two equal factors, extract the [illegible] roots of the first and third terms and connect them by [illegible] of the middle term.*

EXERCISE 34

Resolve into factors:

1. $x^2 - 2xy + y^2$.
2. $a^2 + 4ab + 4b^2$.
3. $4a^2 - 4ab + b^2$.
4. $9a^2 + 24ac + 16c^2$.
5. $a^4 + 2a^2b^2 + b^4$.
6. $4x^2 - 4x^2y + x^2y^2$.
7. $a^4 - 4a^2b^2 + 4b^4$.
8. $9a^4 + 12a^3b + 4a^2b^2$.
9. $x^6 - 2x^3y^3 + y^6$.
10. $a^2x^4 + 4abx^2y^2 + 4b^2y^4$.
11. $25a^4b^4 + 40a^2b^2c^2 + 16c^4$.
12. $49r^4s^2 - 70r^2st^2 + 25t^4$.
13. $1 - 16z^2 + 64z^4$.
14. $9x^2 - 24xyz + 16y^2z^2$.
15. $36a^4b^4 + 60a^2b^2c^2 + 25c^4$.
16. $25u^6v^6 - 10u^3v^3 + 1$.
17. $121a^4 - 286a^2y + 169y^2$.
18. $(a + b)^2 - 4c(a + b) + 4c^2$.
19. $(x - y)^2 + 6a(x - y) + 9a^2$.
20. $25 + 10(c - d) + (c - d)^2$.

61. Case 3.—A Binomial which is the Difference of Two Squares.—Since $(a + b)(a - b) = a^2 - b^2$, the factors of $a^2 - b^2$ are $a + b$ and $a - b$. Any binomial of this form must therefore be the product of the sum and difference of the two numbers of which the two terms of the binomials are the squares. Hence the

Rule.—*The difference of the squares of two numbers is the product of the sum and the difference of the two numbers.*

1. Factor $36a^2x^2 - 49b^2y^6$.

As $36a^2x^2$ is the square of $6ax$ and $49b^2y^6$ is the square of $7by^3$, the factors of $36a^2x^2 - 49b^2y^6$ will be the sum and the difference of the two numbers $6ax$ and $7by^3$. Therefore $36a^2x^2 - 49b^2y^6 = (6ax + 7by^3)(6ax - 7by^3)$.

2. Factor $25u^{10} - 16u^8v^8$.

Although both terms are squares, note that the terms contain the common monomial factor u^8 and this factor should first be separated from the binomial according to the rule of *Case* 1.

$$25u^{10} - 16u^8v^8 = u^8(25u^2 - 16v^8) =$$

Factoring $25u^2 - 16v^8$, we get, $u^8(5u + 4v^4)(5u - 4v^4)$.

3. Factor $(2a - b)^2 - (3b + d)^2$.

As $2a - b$ and $3b + d$ are placed in parentheses, they are treated as single numbers, like monomials, and the expression may be factored by the same rule as any binomial which is the difference of two squares. Hence

$(2a - b)^2 - (3b + d)^2 = [(2a - b) + (3b + d)][(2a - b) - (3b + d)]$

Removing inner parentheses, $= (2a - b + 3b + d)(2a - b - 3b - d)$

Collecting like terms, $= (2a + 2b + d)(2a - 4b - d)$

EXERCISE 35

Resolve into factors:

1. $u^2 - v^2$.
2. $4a^2 - b^2$.
3. $9x^2 - 16y^2$.
4. $a^2b^2 - c^2$.
5. $x^2 - y^2z^2$.
6. $1 - a^2$.
7. $9 - x^4$.
8. $a^6 - 16$.
9. $4 - 9y^8$.
10. $25x^6 - 49y^4$.
11. $1 - 121c^6$.
12. $a^4 - 4b^2$.
13. $x^6 - 4y^4$.
14. $9a^4 - 4a^2b^2$.
15. $25a^4b^4 - 16c^4$.
16. $49r^4s^2 - 25t^4$.
17. $1 - 64z^4$.
18. $25u^6v^6 - u^6$.
19. $121a^4 - 169y^2$.
20. $a^2x^4 - 4a^2y^4$.
21. $4a^3 - ab^2$.
22. $4x^2 - x^2y^2$.
23. $a^4 - b^4$.
24. $1 - 256y^8$.
25. $m^3n^3 - 16mn$.
26. $(a + b)^2 - c^2$.
27. $(a - b)^2 - c^2$.
28. $(x + 2y)^2 - z^2$.
29. $(2x - y)^2 - 9z^2$.
30. $(x - y)^2 - (u + v)^2$.
31. $(x + y)^2 - (u - v)^2$.
32. $(m + 2n)^2 - (p - q)^2$.

33. $(2a + 3b)^2 - 9(c - 2d)^2$.
34. $(3x - 2y)^2 - (2x + 3y)^2$.
35. $(a + 4)^2 - 25(a + 3)^2$.
36. $(a - b)^2 - 1$.
37. $9 - (x - y)^2$.
38. $9x^2 - (x - 3c)^2$.
39. $(a + b - c)^2 - (d + 5)^2$.
40. $(a - b + c)^2 - (b - 2c)^2$.
41. $(x + y + z)^2 - (x - y - z)^2$.
42. $(2x - y + z)^2 - (x + y - 3z)^2$.

62. Case 4. A Trinomial which is the Product of Two Binomials Having First Terms Equal.

In the preceding chapter, a rule was given for writing the product of two binomials, whose first terms are alike.

Thus, $(x + 5)(x - 3) = x^2 + 2x - 15$,
where the sum of the two second terms of the binomials is equal to the coefficient of the middle term and the product of the two second terms is equal to the third term of the trinomial.

Hence for any trinomial of this general form, $x^2 + ax + b$, where a represents the coefficient of x, and b the third term of the trinomial, the rule for finding its factors is simply the reverse of the rule for finding the product.

Rule. *The first term of each factor is the square root of the first term of the trinomial. The second terms of the factors are two numbers, whose product will equal the third term of the trinomial, and whose algebraic sum will equal the coefficient of the middle term.*

1. Factor $x^2 + 11x + 28$.

The first term of each factor is x.

The two second terms of the factors must be two numbers whose product is 28 and whose algebraic sum is 11. These numbers are evidently 4 and 7.

Hence $x^2 + 11x + 28 = (x + 4)(x + 7)$.

2. Factor $x^2 - 9x + 20$.

The first term of each factor is x.

The two second terms are two numbers whose product is 20 and whose sum is -9. As the product is $+20$, the two numbers must have the same sign and as their sum is -9, they must both be minus. Therefore, the two numbers are -4 and -5 and

$$x^2 - 9x + 20 = (x - 4)(x - 5)$$

3. Factor $x^2 + x - 30$.

In this case, since the last term is minus, the two numbers must have different signs, and as their sum must equal $+1$, the positive number is the larger.

Hence, the two numbers are $+6$ and -5, as their product is -30 and their sum, $+1$, and

$$x^2 + x - 30 = (x + 6)(x - 5)$$

4. Factor $x^2 - 5x - 50$.

Here we must find two numbers, whose product is -50 and whose sum is -5. The product being negative, the two numbers must have different signs and their sum being also negative, the larger number must be negative.

Hence $x^2 - 5x - 50 = (x - 10)(x + 5)$.

5. Factor $x^2 - 3xy + 2y^2$.

Note that in this case, the product of the two second terms must equal $2y^2$ and their sum $-3y$, the coefficient of x.

Hence $x^2 - 3xy + 2y^2 = (x - 2y)(x - y)$.

6. Factor $(x - y)^2 + 5(x - y) - 14$.

This expression is of the same general form, $x^2 + ax + b$, and applying the rule, we find that the first term of each binomial is $(x - y)$ and the second terms, two numbers, whose product equals -14 and whose sum is 5.

Hence $(x - y)^2 + 5(x - y) - 14 = [(x - y) + 7][(x - y) - 2]$.

EXERCISE 36

Factor:

1. $x^2 + 5x + 6$.
2. $x^2 - 5x + 6$.
3. $x^2 + x - 6$.
4. $x^2 - x - 6$.
5. $x^2 + 2x - 15$.
6. $x^2 + 8x + 15$.
7. $x^2 - 3x - 28$.
8. $x^2 - 12x + 35$.
9. $x^2 - 4x - 45$.
10. $x^2 + x - 132$.
11. $x^2 + 3xy + 2y^2$.
12. $x^2 + 13xy - 48y^2$.
13. $a^2 - 3a - 10$.
14. $a^2 + 5a - 24$.
15. $x^2 + 4ax + 3a^2$.
16. $x^2 - 5xy + 6y^2$.
17. $a^2 - 3ax - 10x^2$.
18. $1 + 3ab + 2a^2b^2$.

19. $x^4 + 5ax^2 + 6a^2$.
20. $(a + b)^2 + 3(a + b) - 40$.
21. $(x + y)^2 - 15(x + y) + 50$.
22. $(c + 2d)^2 - (c + 2d) - 20$.
23. $a^2b^2 + 19abc + 48c^2$.
24. $a^4x^4 - 21a^2b^3x^2 + 110b^6$.

63. Case 5. The Sum or Difference of Two Cubes. Referring to the preceding chapter, we see that the sum of the cubes of two numbers is divisible by the sum of the numbers and the quotient is the sum of the squares of the numbers minus their product.

$$\frac{x^3 + y^3}{x + y} = x^2 - xy + y^2$$

Therefore, $x^3 + y^3 = (x + y)(x^2 - xy + y^2)$

That is, the divisor and quotient form the factors of the dividend.

Similarly, as $\dfrac{x^3 - y^3}{x - y} = x^2 + xy + y^2$

$$x^3 - y^3 = (x - y)(x^2 + xy + y^2)$$

Hence to factor the **sum of the cubes** of two numbers, we have the following:

Rule.—*The first factor is the sum of the two numbers. The second factor is the sum of the squares of the two numbers minus the product of the two numbers.*

And to factor the **difference of the cubes** of two numbers,

Rule.—*The first factor is the difference of the two numbers. The second factor is the sum of the squares of the two numbers, plus the product of the two numbers.*

1. Factor $8x^3 + 27y^3$.

$8x^3$ is the cube of $2x$ and $27y^3$ is the cube of $3y$. The first factor is the sum of these numbers, $2x + 3y$. The second factor is the sum of the squares of these two numbers minus their product, or $4x^2 - 6xy + 9y^2$.

Hence $8x^3 + 27y^3 = (2x + 3y)(4x^2 - 6xy + 9y^2)$.

2. Factor $64a^3 - 125b^3$.

$64a^3$ is the cube of $4a$ and $125b^3$ is the cube of $5b$.
Hence $64a^3 - 125b^3 = (4a - 5b)(16a^2 + 20ab + 25b^2)$.

3. Factor $1 + 216a^6$.

1 is the cube of 1 and $216a^6$ is the cube of $6a^2$.
Hence $(1 + 216a^6) = (1 + 6a^2)(1 - 6a^2 + 36a^4)$.

4. Factor $(a + b)^3 - (c + d)^3$.

The two numbers here are $a + b$ and $c + d$.
Therefore, $(a + b)^3 - (c + d)^3 =$

$$[(a + b) - (c + d)][(a + b)^2 + (a + b)(c + d) + (c + d)^2]$$

Simplifying these factors $= (a + b - c - d)(a^2 + 2ab + b^2 + ac + ad + bc + bd + c^2 + 2cd + d^2)$

EXERCISE 37

Factor:

1. $x^3 + 8y^3$.
2. $8a^3 - b^3$.
3. $1 - 27x^3$.
4. $8c^3 + 1$.
5. $8x^3 - 27y^3$.
6. $64a^3 + 8b^3$.
7. $x^3y^3 - z^3$.
8. $27a^3b^3 + 8c^3$.
9. $a^6 - b^3$.
10. $x^9 + 8y^6$.
11. $125x^3y^6 - 64$.
12. $216m^3 + 343n^3$.
13. $p^{15} - q^{18}$.
14. $x^{15} - 216x^9$.
15. $(x + y)^3 + 1$.
16. $1 - (x - y)^3$.
17. $8a^3 + (a - b)^3$.
18. $(a - b)^3 - (2a + b)^3$.
19. $8(x - 3)^3 + 125$.
20. $216 - 64(a + 3)^3$.
21. $1 - (x - y)^6$.
22. $40a^5 + 5a^2b^3$.
23. $27m^3n - 216n$.
24. $a^3b^3c^6 - 343$.
25. $56x^7 + 7xy^3$.

64. Case 6. A Polynomial Whose Terms May be Grouped to Show a Common Compound Factor.

If a polynomial can be grouped, so that each group contains a common compound factor, that is, a factor consisting of more than one term, then the expression may be factored in a way similar to *Case* 1.

1. Factor $ax + bx + ay + by$.

Grouping the first two terms and the last two terms,

$$(ax + bx) + (ay + by)$$

Each of these two groups contains a common monomial factor.

$$x(a + b) + y(a + b)$$

These two terms contain a common compound factor of $a + b$. Proceeding, as in the case of a common monomial factor, the second factor of the expression is obtained by dividing each group by the common factor $(a + b)$. Dividing the first group by $a + b$, we get x and dividing the second term, we get y. Hence the second factor of our expression is $x + y$ and the two factors are $(a + b)(x + y)$.

2. Factor $a^2 + ab - ac - bc$.

Grouping the first two terms and the last two, changing the signs of the last two, as the parenthesis is preceded by a minus sign,

$$(a^2 + ab) - (ac + bc) =$$

Factoring each group, $a(a + b) - c(a + b) =$
Factoring the expression, $(a + b)(a - c)$

3. Factor $2m^3 - 3m^2 - 6m + 9$.

Grouping, $(2m^3 - 3m^2) - (6m - 9) =$
Factoring each group, $m^2(2m - 3) - 3(2m - 3) =$
Factoring, the expression, $(2m - 3)(m^2 - 3)$

4. Factor $a^3 - 2a^2 + 2 - a$.

To obtain a common compound factor in each group, the order of the terms must be changed.

$$a^3 - 2a^2 + 2 - a = a^3 - 2a^2 - a + 2 =$$

Grouping, $(a^3 - 2a^2) - (a - 2) =$
Factoring each group, $a^2(a - 2) - (a - 2) =$
Factoring the expression, $(a - 2)(a^2 - 1) =$
Factoring $(a^2 - 1)$, $(a - 2)(a + 1)(a - 1)$

5. Factor $x^2 - 4y^2 + x - 2y$.

Grouping, $(x^2 - 4y^2) + (x - 2y) =$
Factoring the first group, $(x + 2y)(x - 2y) + (x - 2y)$
The common compound factor is $(x - 2y)$, dividing each of the two groups by the common factor, we get the second factor $(x + 2y + 1)$.

Hence
$$\begin{aligned} x^2 - 4y^2 + x - 2y &= (x^2 - 4y^2) + (x - 2y) \\ &= (x + 2y)(x - 2y) + (x - 2y) \\ &= (x - 2y)(x + 2y + 1) \end{aligned}$$

6. Factor $6ax^3 - 10ax^2 - 12ax + 20a$.

As each term contains the monomial factor $2a$, this factor is separated before proceeding as usual.

Thus $6ax^3 - 10ax^2 - 12ax + 20a =$

$$2a(3x^3 - 5x^2 - 6x + 10) =$$
$$2a[(3x^3 - 5x^2) - (6x - 10)] =$$
$$2a[x^2(3x - 5) - 2(3x - 5)] =$$
$$2a(3x - 5)(x^2 - 2)$$

The student should note that grouping the terms does not produce factors. If the terms of the original expression are separated by plus or minus signs, the groups will also be separated by a plus or a minus sign.

Rule.—*Group the terms, so that when each group is factored, a common compound factor is shown. Divide the expression by the common compound factor and the quotient is the other factor.*

EXERCISE 38

Factor:

1. $ac + bc + ad + bd.$
2. $ax - bx + ay - by.$
3. $ax + bx - ay - by.$
4. $ax - bx - ay + by.$
5. $x^2 + xy + xz + yz.$
6. $x^2 + xz - xy - yz.$
7. $m^2 - mn + 3m - 3n.$
8. $3ax - 4ay + 3bx - 4by.$
9. $3a^2 - 4ax - 3a + 4x.$
10. $x^2 + ax - bx - ab.$
11. $x^4 + 2ax^2 - 3bx^2 - 6ab.$
12. $a^3 - ab^2 - a^2b + b^3.$
13. $a^5 + a^2b^3 + a^3b^2 + b^5.$
14. $6x^2 + 3xy - 2xz - yz.$
15. $2m^2 + 3km - 14mn - 21kn.$
16. $7ab + 7ac - 2b^2 - 2bc.$
17. $x^3 + x - x^2 - 1.$
18. $5ab + 10a^2 - 2ac - bc.$
19. $(a - b)^2 - (a - b).$
20. $9x^2 + 4y^4 - 12xy^2 - 16.$
21. $3x^3 - 6x^2 + 3x - 6.$
22. $(m + n)(p + q) - 2p(m + n).$
23. $1 + a - a^2bc - a^3bc.$
24. $x^3 + y^3 + 3xy(x + y).$
25. $2(x^2 - y^2) + 3(x + y)^2.$

65. Case 7. A Polynomial Whose Terms May be Grouped to Form the Difference of Two Squares.

If the terms of a polynomial can be grouped to show the difference of two squares, the factors may be found by the rule of *Case* 3.

1. Factor $a^2 - 4ab + 4b^2 - 9c^2$.

Grouping the first three terms,

$$a^2 - 4ab + 4b^2 - 9c^2 = (a^2 - 4ab + 4b^2) - 9c^2 =$$
$$(a - 2b)^2 - (3c)^2 =$$

Factoring, $(a - 2b + 3c)(a - 2b - 3c)$

2. Factor $4a^2 + 9b^2 - 9c^2 - 4d^2 + 12cd + 12ab$.

Since the first and third terms of the square of a binomial must be plus, it is evident that the terms of the expression which are squares, but have a minus sign before them, must have their signs changed to plus. This is done by placing them in a parenthesis preceded by a minus sign. This gives us the useful hint, that the negative square terms belong to the second group. It is readily seen that $12ab$ belongs to the first group and $12cd$ to the second, being the respective middle terms of these two squares.

Thus $4a^2 + 9b^2 - 9c^2 - 4d^2 + 12cd + 12ab =$

$$(4a^2 + 12ab + 9b^2) - (9c^2 - 12cd + 4d^2) =$$
$$(2a + 3b)^2 - (3c - 2d)^2 =$$
$$[(2a + 3b) + (3c - 2d)][(2a + 3b) - (3c - 2d)] =$$
$$(2a + 3b + 3c - 2d)(2a + 3b - 3c + 2d)$$

EXERCISE 39

Factor:

1. $x^2 + 2xy + y^2 - z^2$.
2. $x^2 - 2xy + y^2 - 4z^2$.
3. $4a^2 + 4ac + c^2 - 4d^2$.
4. $a^2 - b^2 + 2bd - d^2$.
5. $4x^2 - y^2 - 2yz - z^2$.
6. $x^2 - u^2 + y^2 - v^2 - 2xy - 2uv$.
7. $x^2 + y^2 - u^2 - v^2 + 2xy + 2uv$.
8. $m^2 - q^2 + 4n^2 + 4mn - p^2 + 2pq$.
9. $9x^4 - 9c^2 + 6cx - x^2$.
10. $2xy + 9 - x^2 - y^2$.
11. $9b^2 - c^2 - 4d^2 + 4cd + 12ab + 4a^2$.
12. $a^2 - 1 - 2ab + b^2$.
13. $4x^2y^2 - x^2 - y^2 + 2xy + x^4 + 4y^4$.
14. $2x^2y^3 - x^4 + 25 - y^6$.

15. $9a^2 + 4b^4 - 12ab^2 - 25.$
16. $4a^2 - 1 + 2x^2 - x^4.$
17. $4v^2 - 4r^2 + u^2 - s^2 - 4rs - 4uv.$
18. $2yz^2 + 1 - y^2 - z^4 - 2x^2 + x^4.$
19. $2x^2y^3 - 2x^3y^2 + x^6 - x^4 - y^6 + y^4.$
20. $x^2 - 2abc + 4a^2b^2c^2 - c^2 + 4abcx - a^2b^2.$

66. Case 8. A Trinomial Written as the Difference of Two Squares by the Addition and Subtraction of the Necessary Term.

1. Factor $a^4 + a^2 + 1$.

As the first and third terms of this expression are squares, the trinomial would be the square of a binomial, if the middle term were equal to twice the product of the square roots of the first and last terms, that is, $2a^2$. The addition of a^2 to the trinomial makes it a perfect square. But, since the value of the expression should remain unchanged, we must subtract a number equal to the added number.

$$\begin{array}{ll} \text{Thus} & a^4 + a^2 + 1 \\ \text{add and subtract} & \underline{\phantom{a^4 +{}} a^2 \phantom{{}+1} - a^2} \\ \text{equals} & a^4 + 2a^2 + 1 - a^2 \end{array}$$

The first three terms are equal to $(a^2 + 1)^2$, therefore $(a^4 + 2a^2 + 1) - a^2 =$

$$(a^2 + 1)^2 - a^2, \text{ the difference of two squares.}$$

Factoring this expression by Rule of *Case* 3, we get the factors $(a^2 + 1 + a)(a^2 + 1 - a)$.

2. Factor $4x^4 - 13x^2 + 1$.

In order that this trinomial should be a perfect square, the middle term must be either $+4x^2$ or $-4x^2$. To make the middle term $+4x^2$, $17x^2$ must be added, and to make the middle term $-4x^2$, $9x^2$ must be added. As our object is the formation of the difference of two squares, the amount to be added and subtracted must itself be a square. Therefore we select $9x^2$ and not $17x^2$.

$$\begin{array}{ll} & 4x^4 - 13x^2 + 1 \\ \text{Adding,} & \underline{\phantom{4x^4 -{}} 9x^2 \phantom{{}+1} - 9x^2} \\ & 4x^4 - 4x^2 + 1 - 9x^2 \\ \text{Grouping the terms,} & (4x^4 - 4x^2 + 1) - 9x^2 = \\ & (2x^2 - 1)^2 - (3x)^2 = \\ \text{Factoring,} & (2x^2 - 1 + 3x)(2x^2 - 1 - 3x) \end{array}$$

EXERCISE 40

Factor:

1. $x^4 + x^2 + 1$.
5. $a^4x^4 - 5a^2x^2y^2 + 4y^4$.
2. $a^4 + a^2b^2 + b^4$.
6. $25x^4y^4 + 31x^2y^2z^2 + 16z^4$.
3. $a^4 - 5a^2b^2 + 4b^4$.
7. $49r^4s^4 - 54r^2s^2t^2 + 25t^4$.
4. $9a^4 + 3a^2b^2 + 4b^4$.
8. $1 - 25z^2 + 64z^4$.
9. $36a^4b^4 - 4a^2b^2c^2 + 25c^4$.
10. $100x^4 - 229x^2y^2 + 121y^4$.
11. $z^8 + z^4 + 1$.
12. $m^4 + 4n^4$.

67. Case 9. A Trinomial, the Product of Two Binomials Whose First Terms are Unlike.

In the preceding chapter, it was found that the multiplication of two binomials, whose first terms are unlike, $(2x + 3)(3x - 2)$ is practically an ordinary multiplication, that is, the product of the two first terms forms the first term of our product, the product of the two second terms forms the last term of the product and the middle term is the sum of the two partial products obtained by multiplying the first term of the first binomial by the second term of the second binomial and the second term of the first binomial by the first term of the second binomial.

Thus $(\underline{2x + \overline{3)(3}x - 2}) = 6x^2 + 5x - 6$.

An expression of this form may be factored by simply reversing the operation of multiplication.

1. Factor $8a^2 - 10a + 3$.

The first terms of the binomial factors must be factors of the first term, $8a^2$, either $8a$ and a, or $4a$ and $2a$. The second terms of the binomial factors must be factors of the last term, 3, either 3 and 1, or -3 and -1.

Those factors of the first and last term must be selected that will give as the sum of the partial products, the middle term, $-10a$.

By actual trial, we find that, $4a$, $2a$, and -3, -1, will produce the proper middle term,

$$(\underline{4a - \overline{3)(2}a - 1})$$

the partial products, $-4a$ and $-6a$, when added producing $-10a$.

Hence $8a^2 - 10a + 3 = (4a - 3)(2a - 1)$.

It is evident that this method of guessing the factors may become very tedious, if the numbers in the expression are large. ~~Therefore,~~ a formal method of factoring is more desirable. This method has for its object the transformation of the trinomial into an expression that may be recognized as the product of two binomials, whose first terms are alike and which may then be factored by the Rule of *Case* 4.

The first terms of trinomials in Case 4 must be squares, hence the first step of this method is to make the first term a square.

Thus, $8a^2 - 10a + 3$, if multiplied by 8, the coefficient of the first term, becomes

$$64a^2 - 80a + 24.$$

Note, that the value of the expression is temporarily changed.

Since $64a^2$ is the square of $8a$, we substitute a single letter for $8a$, and simplify the form.

Let $z = 8a$.

Substituting z wherever $8a$ occurs in the expression, we get, $z^2 - 10z + 24$, a trinomial having the form of those in Case 4. Applying the rule of Case 4, we have,

$$z^2 - 10z + 24 = (z - 6)(z - 4)$$

Replacing the value of z, the factors become,

$$(8a - 6)(8a - 4).$$

Separating the monomial factors,

$$2(4a - 3)4(2a - 1) = 8(4a - 3)(2a - 1)$$

Since the original expression was multiplied by 8, we now divide by 8, to reduce the expression to its original value, hence

$$8a^2 - 10a + 3 = (4a - 3)(2a - 1)$$

2. Factor, $9x^2 + 6xy - 8y^2$.

Multiply by 9, $81x^2 + 54xy - 72y^2$
Let $z = 9x$, $z^2 + 6yz - 72y^2$
Factoring, $(z + 12y)(z - 6y)$

Substituting $9x$ for z, $(9x + 12y)(9x - 6y)$
Factoring farther, $3(3x + 4y)3(3x - 2y) = 9(3x + 4y)(3x - 2y)$
Dividing by 9, the factors finally are, $(3x + 4y)(3x - 2y)$

EXERCISE 41

Factor:

1. $2x^2 + 13x - 7$.
2. $6x^2 + x - 1$.
3. $2x^2 + 23x - 12$.
4. $8x^2 + 2x - 1$.
5. $15x^2 - 8x + 1$.
6. $6x^2 - 13x + 6$.
7. $2x^2 + 5x + 3$.
8. $4a^2 + 23a - 72$.
9. $60x^2 + 7xy - y^2$.
10. $12a^4 + a^2x^2 - x^4$.
11. $2x^2 + 5xy + 2y^2$.
12. $6a^2 - 7ab - 3b^2$.
13. $15x^2 - 16xy - 15y^2$.
14. $6x^2 - 23xy + 20y^2$.
15. $6a^4 + 17a^2b^2 + 12b^4$.
16. $15x^4 - 31x^2y^2 + 10y^4$.
17. $40c^2 - 49cd - 24d^2$.
18. $6a^2c^2 + 7abcd - 20b^2d^2$.
19. $25x^2 - 75xy + 56y^2$.
20. $3a^4c^4 + 7a^2c^2d - 40d^2$.

68. Case 10. A Binomial which is the Sum or Difference of Two Like Powers.

Rule 5 of the preceding chapter shows how to write the quotient of the sum or difference of two like powers by the sum or difference of the numbers.

To factor such a binomial, it is only necessary to write the divisor as one factor and the quotient as the second factor.

1. Factor $x^5 + y^5$.

According to rule, $\dfrac{x^5 + y^5}{x + y} = x^4 - x^3y + x^2y^2 - xy^3 + y^4$

Therefore, $x^5 + y^5 = (x + y)(x^4 - x^3y + x^2y^2 - xy^3 + y^4)$

2. Factor $x^6 + y^6$.

As the sum of two even powers is not divisible by either the sum or the difference of the numbers, $x^6 + y^6$ must be considered the sum of two cubes, $(x^2)^3 + (y^2)^3$ and is then factored by the same rule as problem 1.

$$x^6 + y^6 = (x^2 + y^2)(x^4 - x^2y^2 + y^4)$$

3. Factor $x^{12} - y^{12}$.

This binomial may be considered the difference of two twelfth powers, the difference of two sixth powers, $(x^2)^6 - (y^2)^6$, the difference of two fourth powers, $(x^3)^4 - (y^3)^4$, the difference of two cubes, $(x^4)^3 - (y^4)^3$ or the difference of two squares, $(x^6)^2 - (y^6)^2$.

All possible factors of such a binomial are most easily obtained by considering it the difference of the lowest powers.

Factoring as the difference of two squares,

$$x^{12} - y^{12} = (x^6 + y^6)(x^6 - y^6)$$

The first factor, the sum of two cubes, the second factor the difference of two squares,

$$(x^6 + y^6)(x^6 - y^6) = (x^2 + y^2)(x^4 - x^2y^2 + y^4)(x^3 + y^3)(x^3 - y^3)$$

The last two factors, being the sum and difference of two cubes, may be factored farther.

$$(x^2 + y^2)(x^4 - x^2y^2 + y^4)(x^3 + y^3)(x^3 - y^3) =$$
$$(x^2 + y^2)(x^4 - x^2y^2 + y^4)(x + y)(x^2 - xy + y^2)(x - y)(x^2 + xy + y^2)$$

which gives us all the prime factors of $x^{12} - y^{12}$.

EXERCISE 42

Factor:

1. $a^4 - b^4$.
2. $x^7 + y^7$.
3. $16 - x^4$.
4. $a^6 - 1$.
5. $a^6 + b^6$.
6. $a^5 + b^5$.
7. $x^6 - y^6$.
8. $1 - x^7$.
9. $a^5 - 1$.
10. $32x^5 + y^5$.
11. $16a^4 - 81b^4$.
12. $x^8 - y^4$.
13. $a^{12} + b^{12}$.
14. $16x^8 - 1$.
15. $1 - c^6$.
16. $1 + x^5$.
17. $x^7 - x$.
18. $2x^{10} - 2y^{10}$.

69. Case 11. Theory of Divisors or Factor Theorem.

If a polynominal in x, as $x^3 - 2x^2 + 4x - 8$ is divided by $x - 2$, the quotient will be $x^2 + 4$.

Therefore $x^3 - 2x^2 + 4x - 8 = (x - 2)(x^2 + 4)$.

When $x = 2$, the factor $x - 2 = 0$ and the product of $x^2 + 4$ by $x - 2$ must also be equal to zero. But since $x^3 - 2x^2 + 4x - 8$ equals this product, it must also be equal to zero when $x = 2$.

Substituting 2 for x,

$$x^3 - 2x^2 + 4x - 8 = 2^3 - 2\cdot2^2 + 4\cdot2 - 8 = 8 - 8 + 8 - 8 = 0.$$

Hence, if $x - 2$ is a factor of $x^3 - 2x^2 + 4x - 8$, this expression must reduce to zero for $x = 2$. The reverse of this statement is also true, if, $x^3 - 2x^2 + 4x - 8$ reduces to zero for $x = 2$, then $x - 2$ is a factor of the expression.

Hence the **Factor Theorem:**

If a polynomial in x is reduced to zero when a particular number n is substituted for x, then $x - n$ is a factor of the polynomial.

1. Factor $x^5 + 4x^2 + 16$.

If this expression contains a factor of the form $x - n$, it is evident that n must be an exact divisor of the last term, 16.

The exact divisors of 16 are, 1, -1, 2, -2, 4, -4, 8, -8, 16, -16. Since our expression contains only positive terms, no plus value could reduce it to zero. Hence we only need to try the negative divisors of 16.

By actual trial, -1 does not reduce the expression to zero. Substituting, $x = -2$,

$$x^5 + 4x^2 + 16 = (-2)^5 + 4(-2)^2 + 16 = -32 + 16 + 16 = 0$$

Hence $x - (-2) = x + 2$ must be a factor.

Dividing by $x + 2$, the quotient is $x^4 - 2x^3 + 4x^2 - 4x + 8$

Therefore, $x^5 + 4x^2 + 16 = (x + 2)(x^4 - 2x^3 + 4x^2 - 4x + 8)$

As no exact divisor of 8 will reduce the second factor to zero, both factors are prime.

2. Factor $4x^3 - 7x + 3$.

The exact divisors of 3 are, 1, -1, 3, -3.

Using $x = 1$,

$$4x^3 - 7x + 3 = 4 - 7 + 3 = 0$$

Hence $x - 1$ is a factor.

Dividing by $x - 1$,

$$4x^3 - 7x + 3 = (x - 1)(4x^2 + 4x - 3)$$

EXERCISE 43

Factor:

1. $x^3 - 26x + 5$.
2. $x^3 - 4x^2 + 5x - 2$.
3. $x^3 - 15x - 4$.
4. $x^3 - 8x + 3$.
5. $x^3 - 3x + 2$.
6. $x^3 - 12x + 16$.
7. $x^3 + x^2 - x - 10$.
8. $x^3 - 3x^2 - 4x + 12$.
9. $x^3 + 4x^2 + 2x - 3$.
10. $x^4 + 3x^3 - 4x^2 + x + 7$.
11. $3x^3 + 2x^2 - 7x + 2$.
12. $2x^3 + x^2 - 6x + 3$.
13. $6x^3 - x^2 - 20x + 12$.
14. $x^3 + 3x^2 - 13x - 15$.

REVIEW EXERCISE VII

Factor:

1. $y - a^4y$.
2. $2b^2m - 3ab^2 + 2bmx - 3abx$.
3. $x^3 - 64y^3$.
4. $8m^2 - 6m - 35$.
5. $a^5 + 27a^2$.
6. $17n^2 + 25n - 18$.
7. $y^4 - 1$.
8. $abx^3 + 3abx^2 - abx - 3ab$.
9. $16x^4 + 8x^2y^2 + y^4$.
10. $a^3 - 9a$.
11. $x^2 - 14x + 49$.
12. $(x - y)^2 - b^2$.
13. $6x^2 - x - 77$.
14. $5x^3 - 125xy^2$.
15. $x^7 - 2x^4 + x$.
16. $1 + 2x^2y^2 + 9x^4y^4$.
17. $1 - a^2 + 4q^2 - 4q$.
18. $6x^2 - 19x + 10$.
19. $cdx^2 + adxy - bcxy - aby^2$.
20. $y^4 + 16y^2z^2 + 64z^4$.
21. $z^2 - 29qz + 100q^2$.
22. $64m^3 - 125t^3$.
23. $ax^2 - 3ax - 4a$.
24. $a^2x^2 - 4ax + 3$.
25. $8 - 27a^3x^3$.
26. $a^2 + b^2 - c^2 - 2ab$.
27. $x^2 - 3ax + 4bx - 12ab$.
28. $6x^4y^4 + 4x^2y^2 - 12x^2y^4$.
29. $m^2 - 3mn - 54n^2$.
30. $4mn - 2xn - 4mx + 8m^2$.
31. $5b^6 - 5$.
32. $9x^4 - (2x - y)^2$.
33. $16 - 4a^2 + m^2 - 4am$.
34. $c^2x - xy + 2c^2 - 2y$.
35. $x^6 - y^6$.
36. $a^4 + a$.
37. $c^4 + c^2d^2 + d^4$.
38. $m^3 - m^2 - 30m$.
39. $a^2 + 2xy - x^2 - y^2$.
40. $3x^2 - 36x + 108$
41. $3ax + 6a^2b - 21axy$.
42. $2x^2y + 4xy + 2y$.
43. $6x^2 + 7x + 2$.
44. $9a^4 + 26a^2b^2 + 25b^4$.
45. $3b^3c^3 + 2b^2c^2 - 6bc^3$.
46. $2ac - bc + 4a^2 - 2ab$.
47. $(x + b)^2 - (3bx - 2y)^2$.
48. $aby^2 - axy + bxy - x^2$.
49. $3a^2 - 24a^5$.
50. $x^4 + 2x^2 + 9$.
51. $m^{12} - n^{12}$.
52. $a^4b - a^3b^2 + a^2b^3 - ab^4$.
53. $r^6 + 64s^3$.
54. $a^{16} - b^8$.
55. $a^2b^2 - 4abx - 4x + 2ab + 4x^2$.
56. $x^3 - 6x^2 + 6x - 5$.
57. $x^3 - 8x^2 + 17x - 10$.
58. $x^3 - 13x + 12$.
59. $3x^3 + 7x^2 - 4$.
60. $3x^3 - 27xy^2 + 18y^3 - 2x^2y$.

CHAPTER VIII

HIGHEST COMMON FACTOR AND LEAST COMMON MULTIPLE

COMMON FACTORS

70. A **common factor** of two or more numbers is a factor contained in each of the numbers and therefore, an exact divisor of each number. This definition holds, whether the numbers are expressed in figures, as in Arithmetic, or expressed in figures and letters, as in Algebra. Integral and rational expressions only, that is, expressions that do not contain fractions or indicated roots, are considered in this chapter.

Thus, $3x$ is a common factor of $12x$ and $15x^2$, and $a + b$ is a common factor of $(a + b)^2$ and $(a + b)(a - b)$.

As in Arithmetic, two or more algebraic numbers are said to be *prime* to each other when they have no common factor except 1.

The **highest common factor** of two or more algebraic numbers is the highest or largest factor common to all the numbers.

Thus, 2 is a common factor of 12 and 16, but 4 is the largest common factor.

Similarly, $3x$ is a common factor of $12x^2$ and $15x^3$, but $3x^2$ is the highest common factor, that is, the factor of *highest degree* contained in each number.

It is convenient to use the abbreviation H. C. F. for highest common factor.

1. Find the H. C. F. of $28x^2y^3$ and $35x^3y^5$.

$$28x^2y^3 = 2 \times 2 \times 7 \times x \times x \times y \times y \times y$$
$$35x^3y^5 = 5 \times 7 \times x \times x \times x \times y \times y \times y \times y \times y$$
$$\therefore \text{the H. C. F.} = 7 \times x \times x \times y \times y \times y = 7x^2y^3$$

2. Find the H. C. F. of $x^3 + y^3$ and $x^2 + 2xy + y^2$.

$x^3 + y^3 = (x + y)(x^2 - xy + y^2)$
$x^2 + 2xy + y^2 = (x + y)(x + y)$
$\therefore$ H. C. F. $= x + y$

3. Find the H. C. F. of $6a^2 - 12ab + 6b^2$; $9a^2 - 9b^2$; $12a^2 + 24ab - 36b^2$.

$6a^2 - 12ab + 6b^2 = 6(a^2 - 2ab + b^2) = 6(a - b)^2$
$9a^2 - 9b^2 = 9(a^2 - b^2) = 9(a + b)(a - b)$
$12a^2 + 24ab - 36b^2 = 12(a^2 + 2ab - 3b^2) = 12(a + 3b)(a - b)$
$\therefore$ H. C. F. $= 3(a - b)$

Hence, **to find the H. C. F. of two or more algebraic expressions,** use the following

Rule.—*Factor each expression.*

The product of all the common factors is the highest common factor.

Note, that each common factor is used the least number of times it occurs in any expression.

EXERCISE 44

Find the H. C. F. of:

1. $12x^3$ and $30x^2$.
2. $2a^3bc$ and a^2b^2c.
3. $50cx^3y^2$, $25cx^4y$, and $50cx^3$.
4. $42ak^5$, $21a^2k^6$ and $28a^3k^3$.
5. $24m^3n^3$, $42a^2m^2n^2y$, and $36m^2n^4$.
6. $2(x - y)^2$ and $4(x - y)^3$.
7. $6c(a + b)^3$ and $12b(a + b)^5$.
8. $10(x + y)(x - y)^2$ and $15a(x + y)^2(x - y)$.
9. $7ab(m + n)^2(m - n)$, $21a^2b^2(m + n)(m - n)^2$ and $14a^3b(m + n)^3(m - n)^3$.
10. $a^2 - 5a$ and $a^2 - 10a + 25$.
11. $m^2 - n^2$ and $m^3 - n^3$.
12. $x^2 + 11x + 28$ and $x^2 - x - 20$.
13. $8x^3 + 27y^3$ and $4x^2 - 6xy + 9y^2$.
14. $4x^4 - 13x^2 + 1$ and $2x^2 - 3x - 1$.
15. $8a^2 - 10a + 3$ and $4a^2 - 1$.
16. $m^2 - n^2$, $m^3 - n^3$, $m^2 - 2mn + n^2$.

17. $1 + 125x^3$, $1 + 10x + 25x^2$, $1 - 25x^2$.

18. $3x^3 - 6x^2 + 3x$, $6x^3 + 9x^2 - 15x$, $3x^3 - 3x$.

19. $3a^5 + 9a^4 - 3a^3$, $7a^2x + 21ax - 7x$, $5a^2y^2 + 15ay^2 - 5y^2$.

20. $18m^3 - 39m^2 + 18m$, $18m^3 - 57m^2 + 30m$, $9m^3 - 15m^2 + 6m$.

COMMON MULTIPLE

71. A **multiple** of any number is the product of that number and any multiplier. Thus, 20 is a multiple of 5, because it may be produced by multiplying 5 by 4, similarly $a^2 - b^2$ is a multiple of $a + b$, because it may be produced by multiplying $a + b$ by $a - b$.

It is evident that a multiple of any number is exactly divisible by that number.

A **common multiple** of two or more numbers is a multiple of each number and is therefore exactly divisible by each of the numbers.

Thus, 24 is a common multiple of 2, 3 and 6, because it is a multiple of each one and of course, exactly divisible by each. Similarly, $20a^2b^3$ is a common multiple of $2ab$, $5a^2b$ and $10ab^2$.

The **least common multiple** of two or more numbers is the least or lowest number that is a multiple of each.

Thus, as above, while 24 is a common multiple of 2, 3 and 6, 6 is their least common multiple, as it is the smallest number which is a multiple of each. Similarly, $10a^2b^2$ is the least common multiple of $2ab$, $5a^2b$ and $10ab^2$.

The abbreviation L. C. M. is used for least or lowest common multiple.

1. Find the L. C. M. of $21a^2b^3$ and $28ab^2$.

Since the L. C. M. must contain each expression, it must contain all the factors of each expression and each factor must be used the greatest number of times it occurs in any of the expressions.

$$21a^2b^3 = 3 \times 7 \times a \times a \times b \times b \times b$$
$$28ab^2 = 2 \times 2 \times 7 \times a \times b \times b$$
$$\therefore \text{L. C. M.} = 3 \times 7 \times 2 \times 2 \times a \times a \times b \times b \times b = 84a^2b^3$$

2. Find the L. C. M. of $3x^4 + 15x^3y - 72x^2y^2$ and $6x^3 - 30x^2y + 36xy^2$.

$$\begin{aligned} 3x^4 + 15x^3y - 72x^2y^2 &= 3x^2(x^2 + 5xy - 24y^2) \\ &= 3x^2(x + 8y)\,(x - 3y) \\ 6x^3 - 30x^2y + 36xy^2 &= 6x(x^2 - 5xy + 6y^2) \\ &= 6x(x - 3y)\,(x - 2y) \end{aligned}$$

$\therefore$ L. C. M. $= 6x^2(x + 8y)(x - 3y)(x - 2y)$

Hence, the following

Rule to find the L. C. M. of two or more expressions,

Factor each expression.

The lowest common multiple is the product of all the different factors, each factor being used the greatest number of times it occurs in any expression.

EXERCISE 45

Find the L. C. M. of:

1. $15a^3$ and $25a^2$.
2. $6x^3yz$ and $8x^2y^2z$.
3. $50am^2n^3$, $25am^4n$ and $50am^3$.
4. $35rs^5$, $21r^2s^6$ and $28r^3s^8$.
5. $24a^3b^3$, $42a^2b^2c^2d$ and $36a^2b^4$.
6. $2(a - b)^2$ and $4(a - b)^3$.
7. $6a(x + y)^3$ and $12b(x + y)$.
8. $5(x + y)(x - y)^2$ and $10(x + y)^3(x - y)$.
9. $21x^2y^2(a + b)^2(a - b)$, $7xy(a + b)(a - b)^2$, and $14x^3y(a + b)^3(a - b)^3$.
10. $x^2 - 4x$ and $x^2 - 8x + 16$.
11. $a^2 + b^2$ and $a^3 + b^3$.
12. $m^2 + 11m + 28$ and $m^2 - m - 20$.
13. $8a^3 - 27b^3$ and $4a^2 + 6ab + 9b^2$.
14. $2y^2 - 3y - 1$ and $4y^4 - 13y^2 + 1$.
15. $4x^2 - 1$ and $8x^3 + 1$.
16. $x^3 - y^3$, $x^2 - y^2$, $x^2 - 2xy + y^2$.
17. $1 + 8x + 16$, $1 + 64x^3$, $1 - 16x^2$.
18. $2a^3 - 4a^2 + 2a$, $4a^3 + 6a^2 - 10a$, $2a^3 - 2a$.

19. $7x^2y + 21xy - 7y$, $5x^2y^2 + 15xy^2 - 5y^2$, $3x^5 + 9x^4 - 3x^3$.

20. $18a^3 - 57a^2 + 30a$, $9a^3 - 15a^2 + 6a$, $18a^3 - 39a^2 + 18a$.

REVIEW EXERCISE VIII

Find H. C. F. and L. C. M. of:

1. $6(x^2 + xy)$, $8(xy - y^2)$, $10(x^2 - y^2)$.
2. $x^2 - 7x - 8$ and $x^2 + 5x + 4$.
3. $x^3 + 1$, $x^3 + x^2 + x + 1$, $x^2 + 4x + 3$.
4. $x^2 - xy$, $x^3 - y^3$, $x^2 + xy + y^2$.
5. $x^4 + x^2 - 6$, $3x^4 + 6x^2 - 24$, $x^4 - 10x^2 + 16$.
6. $a^2 - ab - 2b^2$, $a^3 - 8b^3$, $a^2 - 4ab + 4b^2$, $a^4 - 16b^4$.
7. $7a^3 - 7a$, $4a(a - 1)^2$, $3a^3 + 6a^2 + 3a$.
8. $6x^2 + 5x - 6$, $8x^2 + 10x - 3$, $10x^2 + 9x - 9$.
9. $3x^2 - x$, $9x^2 - 1$, $3x^2 + 14x - 5$.
10. $a^3 - b^3$, $4a - 4b$, $a^2 - 3ab + 2b^2$.
11. $a^2 - 4$, $a^3 - 8$, $a^2 - 4a + 4$, $a^2 - 6a + 8$, $2a^2 - 7a + 6$, $ax^2 - 2x + 4a - 8$.
12. $x^3 - 1$, $2x^2 - x - 1$, $3x^2 - x - 2$.
13. $x^3 - y^3$, $x^2 - y^2$, $ax - ay + bx - by$, $x^2 - 2xy + y^2$, $x^2 - 5xy + 4y^2$.
14. $c^4 - d^4$, $c^5 + d^5$, $c^3 + d^3$, $c^2 + 2cd + d^2$.
15. $8x^3 + 27y^3$, $4x^2 + 12xy + 9y^2$, $4x^2 - 9y^2$.
16. $x^2 + x - 12$ $x^2 - 6x + 9$, $ax^2 - 3ax$, $ax - 3a + bx - 3b$.

CHAPTER IX

FRACTIONS

72. Any fraction may be considered an indicated division, the numerator being the dividend, the denominator, the divisor and the value of the fraction, the quotient. Thus $\frac{a}{b}$ is an indicated division. The numerator a is the dividend, the denominator b, the divisor, and the fraction $\frac{a}{b}$ represents the quotient.

As in arithmetic, the numerator and denominator are called the *terms of the fraction.*

In handling algebraic fractions, the same principles are used as in arithmetic.

Since every fraction is a division, and since the dividend and divisor may be multiplied or divided by the same number without altering the value of the quotient, therefore

If the numerator and denominator of a fraction are both multiplied or divided by the same number, the value of the fraction is not altered.

REDUCTION OF FRACTIONS TO THEIR LOWEST TERMS

73. A fraction is in its *lowest terms,* when the numerator and denominator contain no common factors. Hence,

TO REDUCE A FRACTION TO ITS LOWEST TERMS

Factor numerator and denominator and then divide numerator and denominator by the highest common factor.

1. Reduce to its lowest terms $\frac{42x^2y}{49xy^2}$

The highest common factor of $42x^2y$ and $49xy^2$ is $7xy$. Dividing numerator and denominator by $7xy$, we have the fraction $\frac{6x}{7y}$ in its lowest terms.

The pupil should note that this division is equivalent to the usual cancellation of common factors.

$$\frac{42x^2y}{49xy^2} = \frac{6 \times \cancel{7} \times \cancel{x} \times x \times \cancel{y}}{7 \times \cancel{7} \times \cancel{x} \times y \times \cancel{y}} = \frac{6x}{7y}$$

2. $$\frac{a^3 + b^3}{a^2 + 2ab + b^2} = \frac{(a + b)(a^2 - ab + b^2)}{(a + b)^2} = \frac{a^2 - ab + b^2}{a + b}$$

Here the H. C. F. is $a + b$. Dividing numerator and denominator by $a + b$, gives $\frac{a^2 - ab + b^2}{a + b}$ as the fraction in its lowest terms.

3. $$\frac{x^2 - 7x + 10}{x^2 - 5x + 6} = \frac{(x - 5)(x - 2)}{(x - 3)(x - 2)} = \frac{x - 5}{x - 3}$$

Here we divide by the H. C. F. $x - 2$.

EXERCISE 46

Reduce to lowest terms:

1. $\frac{10xy^2}{15x^2y}$
2. $\frac{6mn^2r}{9m^2n^2r^2}$
3. $\frac{4a^2b^3}{8ab^5}$
4. $\frac{13x^2y}{39xz^2}$
5. $\frac{28a^2bc^3}{21a^3b^2c^4}$
6. $\frac{22m^2n^2p}{33mn^3p^2}$
7. $\frac{42a^2m^2n^2y}{24m^3n^3}$
8. $\frac{4(x - y)^3}{2(x - y)^2}$
9. $\frac{7ab(m + n)^2(m - n)}{21a^2b^2(m + n)(m - n)^2}$
10. $\frac{a^2 - 5a}{a^2 - 10a}$
11. $\frac{m^2 - n^2}{m^3 - n^3}$
12. $\frac{x^2 - x - 20}{x^2 + 11x + 28}$
13. $\frac{8x^3 + 27y^3}{4x^2 - 6xy + 9y^2}$
14. $\frac{4x^4 - 13x^2 + 1}{2x^2 - 3x - 1}$
15. $\frac{4a^2 - 1}{8a^2 - 10a + 3}$
16. $\frac{m^3 - n^3}{m^2 - 2mn + n^2}$
17. $\frac{1 + 10x + 25x^2}{1 + 125x^3}$
18. $\frac{3x^3 - 3x}{6x^3 + 9x^2 - 15x}$
19. $\frac{7a^2x + 21ax - 7x}{5a^2y^2 + 15ay^2 - 5y^2}$
20. $\frac{18m^3 - 57m^2 + 30m}{9m^3 - 15m^2 + 6m}$

74. In every division there are three signs to be considered, the sign of the dividend, of the divisor and of the quotient. Similarly, there are three signs to consider in every fraction, the sign of the numerator, of the denominator and of the fraction itself, that is, the sign before the fraction.

It is evident that changing the sign of the dividend and of the divisor will have no effect on the quotient, hence,

Changing the sign of the numerator and of the denominator of a fraction will not alter the value of the fraction.

If the numerator and denominator are compound expressions, the signs of these expressions are changed by changing the sign of every term in the expressions.

In a division, if we change the sign of the dividend or of the divisor, but not of both, the sign of the quotient is changed, hence, to restore the quotient to its previous value, the sign of the quotient must also be changed. Therefore, *to change the sign of the numerator or of the denominator, but not of both, change the sign before the fraction.*

If an algebraic expression is the product of any number of factors, it is evident, from the law of signs, that the signs of an *even number of factors* may be changed without altering the value of the product, but if the signs of an *odd number of factors* are changed, the sign of the product is changed. Hence, changing the signs of an odd number of factors of either the numerator or denominator of a fraction is equivalent to changing the sign of the numerator or denominator and therefore requires a change in the sign before the fraction.

These principles are applied in the following problems.

1. Reduce to lowest terms $\dfrac{4xy(y^2 - x^2)}{5(x^2y - xy^2)}$.

Factoring numerator and denominator,

$$\frac{4xy(y + x)(y - x)}{5xy(x - y)}$$

If the signs of the factor $y - x$ are changed, numerator and denominator will contain the common factor $x - y$. Changing the signs of an odd number of factors of the numerator changes the sign of the numerator and hence requires changing the sign before the fraction.

$$\frac{4xy(y+x)(y-x)}{5xy(x-y)} = -\frac{4xy(y+x)(x-y)}{5xy(x-y)}$$

Dividing numerator and denominator by the H. C. F. $xy(x - y$, we have

$$-\frac{4(x+y)}{5}.$$

2. Reduce to lowest terms, $\dfrac{2b^2 - 2ab}{a^2 - 2ab + b^2}$.

Factoring, $\dfrac{2b(b-a)}{(a-b)^2} = \dfrac{2b(b-a)}{(b-a)^2} = \dfrac{2b}{b-a}$

Here we change the signs of both factors of the denominator and, as the change in an even number of factors does not change the sign of the denominator, the sign before the fraction remains unchanged.

3. Reduce to lowest terms, $\dfrac{9 - 3x}{x^2 - 7x + 12}$.

Factoring, $\dfrac{3(3-x)}{(x-3)(x-4)} = -\dfrac{3(x-3)}{(x-3)(x-4)}$

$$= -\frac{3}{x-4} = \frac{3}{4-x}$$

The sign of the factor $3 - x$ in the numerator is changed, hence the sign before the fraction is also changed. As the denominator contains a negative term, -4, the sign of the denominator may be changed and the sign of the fraction changed to plus.

EXERCISE 47

Reduce to lowest terms:

1. $\dfrac{x^2 - 2xy + y^2}{y^2 - x^2}$.

2. $\dfrac{ab - 3b^2}{27b^3 - a^3}$.

3. $\dfrac{a^2 + 7a - 30}{12 - 7a - a^2}$.

4. $\dfrac{x(a - b)^3}{(b^2 - a^2)(b - a)}$.

5, $\dfrac{1 - x^2}{3x(x - 1)}$.

6. $\dfrac{z^2 - (x + y)^2}{x^2 + xy - xz}$.

7. $\dfrac{a^2 + 5a - 14}{4 - a^2}$.

8. $\dfrac{5xy(x^2 - y^2)}{7(xy^2 - x^2y)}$.

9. $\dfrac{xz - yz - ax + ay}{a^2 - z^2}$.

10. $\dfrac{(x + y + z)^2 - a^2}{(a - x)^2 - (y + z)^2}$.

11. $\dfrac{a^2 - (x - y)^2}{(y - a)^2 - x^2}$.

12. $\dfrac{4x^2 - 4xy + y^2}{3(y - 2x)}$.

75. A mixed expression, corresponding to a mixed number in arithmetic, consists of an integral expression and a fraction. As, $a + \frac{b}{c}$ and $\frac{x}{y} - x^2$.

REDUCTION OF MIXED EXPRESSIONS TO FRACTIONS

In arithmetic, a mixed number is reduced to a fraction by multiplying the whole number by the denominator of the fraction, adding the numerator and placing the sum over the denominator. The value of the whole number is not changed, because it is multiplied and divided by the same number, that is, the denominator of the fraction.

In algebra, the same rule is used.

Multiply the integral expression by the denominator, add the numerator to this product and place the denominator under the sum.

Note, that in this rule, addition is an algebraic addition and includes subtraction, that is, if the sign before the fraction is negative, the terms of the numerator must be subtracted by changing their signs.

1. Reduce to a fraction, $a + x - \frac{a^2 - b^2}{x}$.

$$a + x - \frac{a^2 - b^2}{x} = \frac{x(a + x) - (a^2 - b^2)}{x} = \frac{ax + x^2 - a^2 + b^2}{x}$$

The integral expression, $a + x$, is multiplied by the denominator x, the numerator $a^2 - b^2$ is subtracted from this product and the result placed over the denominator x.

2. Reduce to a fraction, $\frac{x + y}{x - y} - 2$.

$$\frac{x + y}{x - y} - 2 = \frac{x + y - 2(x - y)}{x - y} = \frac{x + y - 2x + 2y}{x - y} = \frac{-x + 3y}{x - y}$$

The same rule is used, whether the integral expression precedes or follows the fraction.

EXERCISE 48

Reduce to a fraction:

1. $a - 2x + \frac{a - 3x}{4}$.	6. $a + b - \frac{b^3}{a^2 - ab + b^2}$.
2. $\frac{a^2 + x^2}{a + x} + a + x$.	7. $\frac{x - 3y}{5} - y + 2x$.
3. $a + x - \frac{a^2 + x^2}{a - x}$.	8. $3x + 2y + \frac{x^2 - y^2}{2x - 3y}$.
4. $\frac{x^3}{a - x} - (a^2 + ax + x^2)$.	9. $2a - 5 - \frac{11 - 4a}{a - 3}$.
5. $a + 4 - \frac{a - 12}{a + 3}$.	10. $m + n + \frac{m^2 + n^2}{m - n}$.

REDUCTION OF FRACTIONS TO MIXED EXPRESSIONS

76. An improper fraction, in arithmetic, is changed to a mixed number by dividing the numerator by the denominator, using the remainder as the numerator of the remaining fraction.

Similarly in algebra, if the numerator of a fraction is of a higher degree than the denominator, it may be changed to a mixed expression by the method of division.

To change a fraction to a mixed expression, *divide the numerator by the denominator. The quotient will be the integral expression and the remainder becomes the numerator of the remaining fraction.*

1. Change to a mixed expression, $\frac{5x^2 - 10x + 3}{5x}$

Dividing the numerator by $5x$,

$$\frac{5x^2 - 10x + 3}{5x} = x - 2 + \frac{3}{5x}$$

Note that the remaining fraction must be connected with the integral expression by the sign +.

2. Change to a mixed expression $\dfrac{x^3 - y^3}{x + y}$

$$\begin{array}{l}
x^3 - y^3 \quad)\underline{x \ + y} \\
\underline{x^3 + x^2y} \quad)\underline{x^2 - xy + y^2} \\
\quad - x^2y - y^3 \\
\quad \underline{- x^2y - xy^2} \\
\qquad\qquad xy^2 - y^3 \\
\qquad\qquad \underline{xy^2 + y^3} \\
\qquad\qquad\quad - 2y^3
\end{array}$$

Hence $\dfrac{x^3 - y^3}{x + y} = x^2 - xy + y^2 + \dfrac{-2y^3}{x + y} = x^2 - xy + y^2 - \dfrac{2y^3}{x + y}$

By changing the sign before the fraction, the sign of the numerator may be changed to +.

EXERCISE 49

Reduce to a mixed expression:

1. $\dfrac{6a^2 - 15a + 4}{3a}$.
2. $\dfrac{4x^2 - 28x - 12}{2x}$.
3. $\dfrac{x^3 + y^3}{x - y}$.
4. $\dfrac{a^3 - b^3}{a + b}$.
5. $\dfrac{x^2 + 2xy + 2y^2}{x + y}$.
6. $\dfrac{4a^2 - 4ab - 2b^2}{2a - b}$.
7. $\dfrac{x^3 + 2y^3}{x + 2y}$.
8. $\dfrac{4a^3 - 3a^2 + 10a - 2}{a^2 + a - 3}$.
9. $\dfrac{6a^2 + 4ab - 27b^2 - 4}{3a - 4b}$.
10. $\dfrac{10x^3 - 9x^2 + 6}{5x^2 + 3x - 2}$.

REDUCTION OF FRACTIONS TO EQUIVALENT FRACTIONS HAVING THE LOWEST COMMON DENOMINATOR

77. In order to be able to add fractions, they must have equal or common denominators. As in arithmetic, so in algebra, the least common multiple of all the denominators is used as the lowest common denominator. As the value of each fraction must remain unchanged, numerator and denominator of each fraction are multiplied by the factor necessary to change each denominator to the lowest common denominator.

1. Reduce to equivalent fractions having the lowest common denominator,

$$\frac{3x}{4a^2}, \frac{2x^2}{3a}, \frac{5}{6a^3}.$$

The L. C. M. of the denominators is $12a^3$.

To change the first denominator $4a^2$ to the L. C. D. $12a^3$, it must be multiplied by the missing factor $3a$. Hence, the first numerator must also be multiplied by $3a$, giving us $\frac{9ax}{12a^3}$. Similarly, numerator and denominator of the second fraction are multiplied by the missing factor $4a^2$, producing the fraction $\frac{8a^2x^2}{12x^3}$, and the last fraction becomes equal to $\frac{10}{12a^3}$ when numerator and denominator are multiplied by 2, the factor missing from the denominator $6a^3$.

2. Reduce to equivalent fractions having the lowest common denominator,

$$\frac{3x}{x^2+x-6}, \frac{4x}{x^2-x-12}, \frac{2x}{x^2-6x+8}.$$

Factoring the denominators,

$$\frac{3x}{(x+3)(x-2)}, \frac{4x}{(x+3)(x-4)}, \frac{2x}{(x-4)(x-2)}.$$

The L. C. D. is $(x+3)(x-2)(x-4)$.

Multiplying numerator and denominator of each fraction by the factor missing from the respective denominator,

$$\frac{3x(x-4)}{(x+3)(x-2)(x-4)}, \frac{4x(x-2)}{(x+3)(x-4)(x-2)}, \frac{2x(x+3)}{(x+3)(x-4)(x-2)}$$

Hence the following **Rule:**

To reduce fractions to equivalent fractions having the lowest common denominator, find the L. C. M. *of all the denominators for the lowest common denominator. Then multiply numerator and denominator by the factor missing from the respective denominator.*

Note that all fractions should be reduced to their lowest terms before finding the lowest common denominator.

EXERCISE 50

Reduce to equivalent fractions having the lowest common denominator:

1. $\frac{3b}{2ax}, \frac{8ay}{3a^2b}, \frac{2a}{5b^2x^2}$.

2. $\frac{2a-5}{3}, \frac{5a-2}{4}, \frac{2+3a}{5}$.

3. $\frac{2x-y}{5a}, \frac{2y^2-3x}{3ay}, \frac{4x^2+y}{2a^2y^2}$.

4. $\frac{1}{a+1}, \frac{1}{a+2}$.

5. $\frac{x}{a-4}, \frac{3x}{a+3}$.

6. $\frac{x}{x-y}, \frac{z}{z-y}$.

7. $\frac{2}{1-a}, \frac{5}{1-a^2}$.

8. $\frac{a}{x-y}, \frac{b}{x^2-y^2}$.

9. $\frac{1}{1-4y^2}, \frac{3}{1+2y}$.

10. $\frac{a}{x^3-27}, \frac{a^2}{x-3}$.

11. $\frac{x^2+4y^2}{x^2-4y^2}, \frac{x-2y}{x+2y}$.

12. $\frac{m^2+n^2}{16m^2-9n^2}, \frac{m}{4m-3n}$.

13. $\frac{r+s}{r-s}, \frac{r-s}{r+s}$.

14. $\frac{3a^2}{4(x+y)}, \frac{5a}{3(x^2-y^2)}$.

15. $\frac{p+q}{4r}, 1, \frac{p-q}{3r^2}$.

16. $\frac{m+n}{p-2q}, \frac{m-n}{2p-4q}, \frac{m^2-n^2}{3p-6q}$.

17. $\frac{a}{1-a^3}, \frac{a^2}{1-a^2}, \frac{a^3}{1-a}$.

18. $\frac{1}{(a-b)(a-c)}, \frac{1}{(a-b)(b-c)}, \frac{1}{(a-c)(b-c)}$.

19. $\frac{x}{1+a}, \frac{y}{(1+a)^2}, \frac{z}{(1+a)^3}$.

20. $\frac{x+3}{x^2+x-6}, \frac{x+3}{x^2-6x+8}, \frac{x+3}{x^2-2x-8}$.

ADDITION AND SUBTRACTION OF FRACTIONS

78. Algebraic addition includes subtraction, hence to find the algebraic sum of any number of fractions, it is only necessary to change them to a common denominator and then add or subtract the numerators as shown by the sign before each fraction.

The highest common factor of $42x^2y$ and $49xy^2$ is $7xy$. Dividing numerator and denominator by $7xy$, we have the fraction $\frac{6x}{7y}$ in its lowest terms.

The pupil should note that this division is equivalent to the usual cancellation of common factors.

$$\frac{42x^2y}{49xy^2} = \frac{6 \times \cancel{7} \times \cancel{x} \times x \times \cancel{y}}{7 \times \cancel{7} \times \cancel{x} \times y \times \cancel{y}} = \frac{6x}{7y}$$

2. $$\frac{a^3 + b^3}{a^2 + 2ab + b^2} = \frac{(a + b)(a^2 - ab + b^2)}{(a + b)^2} = \frac{a^2 - ab + b^2}{a + b}$$

Here the H. C. F. is $a + b$. Dividing numerator and denominator by $a + b$, gives $\frac{a^2 - ab + b^2}{a + b}$ as the fraction in its lowest terms.

3. $$\frac{x^2 - 7x + 10}{x^2 - 5x + 6} = \frac{(x - 5)(x - 2)}{(x - 3)(x - 2)} = \frac{x - 5}{x - 3}$$

Here we divide by the H. C. F. $x - 2$.

EXERCISE 46

Reduce to lowest terms:

1. $\dfrac{10xy^2}{15x^2y}$
2. $\dfrac{6mn^2r}{9m^2n^2r^2}$
3. $\dfrac{4a^2b^3}{8ab^5}$
4. $\dfrac{13x^2y}{39xz^2}$
5. $\dfrac{28a^2bc^3}{21a^3b^2c^4}$
6. $\dfrac{22m^2n^2p}{33mn^3p^2}$
7. $\dfrac{42a^2m^2n^2y}{24m^3n^3}$
8. $\dfrac{4(x - y)^3}{2(x - y)^2}$
9. $\dfrac{7ab(m + n)^2(m - n)}{21a^2b^2(m + n)(m - n)^2}$
10. $\dfrac{a^2 - 5a}{a^2 - 10a + 25}$
11. $\dfrac{m^2 - n^2}{m^3 - n^3}$
12. $\dfrac{x^2 - x - 20}{x^2 + 11x + 28}$
13. $\dfrac{8x^3 + 27y^3}{4x^2 - 6xy + 9y^2}$
14. $\dfrac{4x^4 - 13x^2 + 1}{2x^2 - 3x - 1}$
15. $\dfrac{4a^2 - 1}{8a^2 - 10a + 3}$
16. $\dfrac{m^3 - n^3}{m^2 - 2mn + n^2}$
17. $\dfrac{1 + 10x + 25x^2}{1 + 125x^3}$
18. $\dfrac{3x^3 - 3x}{6x^3 + 9x^2 - 15x}$
19. $\dfrac{7a^2x + 21ax - 7x}{5a^2y^2 + 15ay^2 - 5y^2}$
20. $\dfrac{18m^3 - 57m^2 + 30m}{9m^3 - 15m^2 + 6m}$

74. In every division there are three signs to be considered, the sign of the dividend, of the divisor and of the quotient. Similarly, there are three signs to consider in every fraction, the sign of the numerator, of the denominator and of the fraction itself, that is, the sign before the fraction.

It is evident that changing the sign of the dividend and of the divisor will have no effect on the quotient, hence,

Changing the sign of the numerator and of the denominator of a fraction will not alter the value of the fraction.

If the numerator and denominator are compound expressions, the signs of these expressions are changed by changing the sign of every term in the expressions.

In a division, if we change the sign of the dividend or of the divisor, but not of both, the sign of the quotient is changed, hence, to restore the quotient to its previous value, the sign of the quotient must also be changed. Therefore, *to change the sign of the numerator or of the denominator, but not of both, change the sign before the fraction.*

If an algebraic expression is the product of any number of factors, it is evident, from the law of signs, that the signs of an *even number of factors* may be changed without altering the value of the product, but if the signs of an *odd number of factors* are changed, the sign of the product is changed. Hence, changing the signs of an odd number of factors of either the numerator or denominator of a fraction is equivalent to changing the sign of the numerator or denominator and therefore requires a change in the sign before the fraction.

These principles are applied in the following problems.

1. Reduce to lowest terms $\frac{4xy(y^2 - x^2)}{5(x^2y - xy^2)}$.

Factoring numerator and denominator,

$$\frac{4xy(y + x)(y - x)}{5xy(x - y)}$$

If the signs of the factor $y - x$ are changed, numerator and denominator will contain the common factor $x - y$. Changing the signs of an odd number of factors of the numerator changes the

10. $\dfrac{a}{a+b}+\dfrac{b}{a-b}.$ 11. $\dfrac{x+3}{x-y}-\dfrac{4}{x^2-2xy+y^2}.$

12. $\dfrac{a+b}{b-a}+\dfrac{a-b}{(a+b)^2}-\dfrac{a}{a^2-b^2}.$

13. $\dfrac{mr}{m-1}+\dfrac{d}{1+n}.$

14. $\dfrac{v}{u-v}-\dfrac{v}{u+v}+\dfrac{1}{u^2-v^2}.$

15. $\dfrac{1}{x}+\dfrac{1}{y}-\dfrac{1}{x-y}+\dfrac{1}{x+y}.$

16. $\dfrac{x^2+4xy}{x^3+y^3}+\dfrac{1}{x+y}-\dfrac{x}{x^2-xy+y^2}.$

17. $\dfrac{a+b}{a-b}+\dfrac{a-b}{a+b}-2.$

18. $\dfrac{x^2-9y^2}{x^2+6xy+9y^2}-\dfrac{x^2-6xy}{x^2-9y^2}.$

19. $\dfrac{2}{x-7}-\dfrac{x}{14+5x-x^2}-\dfrac{x}{x^2-9x+14}.$

20. $\dfrac{y^2}{y^2-1}+\dfrac{y}{y+1}+\dfrac{y}{1-y}.$

21. $\dfrac{2}{y^2+8y+16}-\dfrac{1}{y(y+4)}+\dfrac{4}{y^2(y+4)}.$

22. $\dfrac{1}{(x-y)(x-z)}-\dfrac{1}{(x-y)(z-y)}-\dfrac{1}{(z-x)(y-z)}.$

23. $\dfrac{1}{a^2+11a+30}-\dfrac{1}{a^2-36}+\dfrac{1}{a^2-25}.$

24. $\dfrac{x+2}{x^2-2x-8}-\dfrac{x+2}{x^2-x-6}+\dfrac{x-4}{x^2-7x+12}.$

25. $\dfrac{1}{(x-1)(x-2)}+\dfrac{1}{(x-2)(3-x)}+\dfrac{1}{(x-3)(1-x)}.$

26. $\dfrac{mn}{(n-m)(m-q)}+\dfrac{mq}{(m-n)(n-q)}+\dfrac{nq}{(n-q)(q-m)}.$

27. $\dfrac{y+z}{(x-y)(x-z)}+\dfrac{x+z}{(y-z)(y-x)}+\dfrac{x+y}{(z-x(z-y)}.$

28. $\dfrac{1}{xyz}-\dfrac{1}{x(x-y)(x-z)}+\dfrac{1}{yz(y-x)(y-z)}.$

29. $\dfrac{m^2-pq}{(m-n)(m-q)}+\dfrac{n^2+mq}{(n-m)(n+q)}+\dfrac{q^2+mn}{(q-m)(q+n)}.$

30. $\dfrac{4}{x^3-y^3}+\dfrac{3x}{y^2-x^2}-\dfrac{2x^2}{(y-x)^2}.$

MULTIPLICATION OF FRACTIONS

79. Algebraic fractions are multiplied in the same manner as arithmetical fractions. Hence,

To Find the Product of Two or More Fractions,

Find the product of the numerators for the numerator of the product, and the product of the denominators for the denominator of the product.

1. Find the product of $\dfrac{3xy^2}{4ab}\times\dfrac{8a^2b}{15x^2y}\times\dfrac{10x^3y^3}{9a^2b^2}.$

Cancelling every factor common to any numerator and denominator

$$\frac{\overset{y}{\cancel{3}\cancel{x}\cancel{y^2}}}{\cancel{4}\cancel{a}\cancel{b}}\times\frac{\overset{2}{\cancel{8}\cancel{a^2}\cancel{b}}}{\underset{\cancel{5}\cancel{x}}{\cancel{1}\cancel{5}\cancel{x^2}\cancel{y}}}\times\frac{\overset{2x^2}{\cancel{1}\cancel{0}\cancel{x^3}y^3}}{9\cancel{a^2}b^2}=\frac{4x^2y^4}{9b^2}.$$

2. Find the product of

$$\frac{a^3-b^3}{a^2+2ab+b^2}\times\frac{a^2+ab^2}{a^2-b^2}\times\frac{ab+b^2}{a^2+ab+b^2}.$$

Factoring all numerators and denominators,

$$\frac{(a-b)(a^2+ab+b^2)}{(a+b)^2}\times\frac{a(a+b)}{(a+b)(a-b)}\times\frac{b(a+b)}{a^2+ab+b^2}.$$

Cancelling common factors of numerators and denominators, and taking the products of the remaining factors, the product is $\dfrac{ab}{a+b}.$

3. Find the product of $\left(1-\dfrac{y^2}{x^2}\right)\left(\dfrac{x^3}{x+y}\right).$

Changing the mixed expression to a fraction,

$$\frac{x^2-y^2}{x^2}\times\frac{x^3}{x+y}=\frac{(x+y)(x-y)}{x^2}\times\frac{x^3}{x+y}=\frac{x(x-y)}{1}=x(x-y).$$

EXERCISE 52

Find the product of:

1. $\dfrac{4y^2}{x^2} \times \dfrac{4x^2}{3y^2} \times \dfrac{2x}{3y}$.

2. $\dfrac{3x^2y^2z^2}{4mnr} \times \dfrac{10m^2n^2r^3}{15x^3y^3z^3}$.

3. $\dfrac{2abc}{5x^2y} \times \dfrac{4abx^3}{10a^2b^2y^3} \times \dfrac{25a^2bx^2y^2}{32abxy}$.

4. $\dfrac{2}{x+2} \times \dfrac{x^2-4}{10}$.

5. $\dfrac{a-b}{a+b} \times \dfrac{a^2-b^2}{a-b}$.

6. $(x+y) \times \dfrac{1}{x^3+y^3}$.

7. $\dfrac{a^2+2ab+b^2}{a^3-b^3} \times \dfrac{a-b}{a^2-b^2}$.

8. $(m^2+mn+n^2) \times \dfrac{m+n}{m^3-n^3}$.

9. $\dfrac{3a(a+3b)}{a^3+b^3} \times \dfrac{a^2-ab+b^2}{a^2-9b^2}$.

10. $\dfrac{m+n}{m^4-n^4} \times \dfrac{m^2-n^2}{(m+n)^2}$.

11. $\dfrac{x+y}{x^6-y^6} \times \dfrac{x^3+y^3}{x^2-y^2}$.

12. $\dfrac{3a(a-b)}{5b(a+b)} \times \dfrac{15(a^2-b^2)}{9(a-b)^2}$.

13. $\dfrac{16c^3b}{7(c^3-b^3)} \times \dfrac{35(c^2+cb+b^2)}{14c^2b^2}$.

14. $\dfrac{x^2+3x+2}{x^2-5x+6} \times \dfrac{x^2-7x+12}{x^2+8x+7}$.

15. $\dfrac{a^2-a}{a^2-1} \times \dfrac{2a^2-4a+2}{4a^2+6a}$.

16. $\dfrac{a^2+a-2}{a^2-3a} \times \dfrac{a^2+9a-36}{a^3+2a^2}$.

17. $\dfrac{a^2-6a-16}{a^2+4a-21} \times \dfrac{a^2-8a+15}{a^2+9a+14}$.

18. $\dfrac{a^2-1}{a^2-4a-5} \times \dfrac{a^2-25}{a^2+2a-3}$.

19. $\dfrac{a^2+9ab+18b^2}{a^2-9ab+20b^2} \times \dfrac{ab^2-4b^3}{a^2+6ab+9b^2}$.

20. $\dfrac{m^2+4m-32}{m^2-8m+15} \times \dfrac{m^2-10m+21}{m^2+m-56}$.

21. $\dfrac{x^2+2x-15}{x^2-9} \times \dfrac{x^2-11x+30}{x^3-6x^2+9x} \times \dfrac{x^2-3x}{x^2-25}$.

22. $\dfrac{y(x-y)}{x^2+2xy+y^2} \times \dfrac{x^2-y^2}{xy^2} \times \dfrac{x^2-y^2}{xy}.$

23. $\dfrac{5a^2+9a-2}{7a^2+17a-12} \times \dfrac{6a^2-11a-10}{3a^2+2a-5} \times \dfrac{21a^2+23a-20}{10a^2-27a+5}.$

24. $\dfrac{7r^4-7r^3}{7r^2-56r-63} \times \dfrac{r^2+r-90}{r^4-9r^3+8r^2}.$

25. $\dfrac{a^2-b^2+2bc-c^2}{x+y+z} \times \dfrac{x^2+y^2+2xy-z^2}{a+b-c}.$

26. $\dfrac{3x^2-5x+2}{4x^2-5x+1} \times \dfrac{4x^2-17x+4}{6x^2-7x+2} \times \dfrac{10x^2-21x+9}{5x^2-23x+12}.$

27. $\left(x-\dfrac{y^2}{x}\right)\left(y+\dfrac{x^2-y^2}{y}\right).$

28. $\dfrac{ac+bc-ad-bd}{xy-4x-3y+12} \times (xy-4x+3y-12).$

29. $\dfrac{(a+b)^2-c^2}{a^2+ab-ac} \times \dfrac{a}{(a-c)^2-b^2} \times \dfrac{(a-b)^2-c^2}{ab-b^2+bc}.$

30. $\dfrac{6t^2-7t+2}{10t^2-7t+1} \times \dfrac{6t^2-5t+1}{6t^2+t-1} \times \dfrac{10t^2+3t-1}{5t^2-4t-1}.$

DIVISION OF FRACTIONS

80. Fractions are divided by the same rule as in arithmetic.

Thus, $\dfrac{5}{7} \div \dfrac{15}{14} = \dfrac{5}{7} \times \dfrac{14}{15} = \dfrac{2}{3}.$

In the same way, $\dfrac{a}{b} \div \dfrac{c}{d} = \dfrac{a}{b} \times \dfrac{d}{c} = \dfrac{ad}{bc}.$

Hence, **To Divide by a Fraction,**

Invert the divisor and proceed as in multiplication.

As in multiplication of fractions, all mixed expressions or integral expressions are changed to fractions.

1. Find the quotient of,

$$\frac{x^2}{y^2} \div \frac{x}{y} = \frac{x^2}{y^2} \times \frac{y}{x} = \frac{x}{y}.$$

2. Find the quotient of

$$\frac{ax - ay}{b(x + y)} \div (x - y) = \frac{a(x - y)}{b(x + y)} \div \frac{x - y}{1} = \frac{a(x - y)}{b(x + y)} \times \frac{1}{x - y} = \frac{a}{b(x + y)}.$$

3. Find the quotient of

$$\left(x - \frac{y^2}{x}\right) \div (x^2 - y^2) = \frac{x^2 - y^2}{x} \div \frac{(x + y)(x - y)}{1} = \frac{(x + y)(x - y)}{x} \times \frac{1}{(x + y)(x - y)} = \frac{1}{x}$$

EXERCISE 53

Find the quotient of:

1. $\dfrac{ab}{c} \div \dfrac{a}{c}$.
2. $\dfrac{5ab}{2cx} \div \dfrac{10a}{4c^2x^2}$.
3. $\dfrac{abc}{xy} \div \dfrac{ac}{bx}$.
4. $\dfrac{cx^2}{y^2z} \div \dfrac{dx}{yz}$.
5. $\dfrac{abc}{cdf} \div \dfrac{ab}{cd}$.
6. $\dfrac{xyz}{rs} \div \dfrac{x^2y}{rs^2}$.
7. $\dfrac{1}{x} \times \dfrac{ab}{xy} \div \dfrac{b}{x^2}$.
8. $\dfrac{x^2 - 4}{x^2 - 1} \div \dfrac{x - 2}{x + 1}$.
9. $\dfrac{a^3 + 1}{a^3 - a^2 + a} \div \dfrac{a + 1}{a - 1}$.
10. $\dfrac{x^2 - 4xy + 3y^2}{x^2 - y^2} \div \dfrac{(x - y)^2}{x^2 - xy}$.
11. $\dfrac{(a + 1)^2}{a} \times \dfrac{(a - 1)^2}{a} \div \dfrac{(a^2 - 1)^2}{ab}$.
12. $\dfrac{m - 2}{m^3 + 8} \div (m^2 + 2m - 4)$.
13. $\dfrac{a^3 + 8}{a^3 - 8} \div \dfrac{a^2 - 4}{a^2 + 2a + 4}$.
14. $\dfrac{7(c - 1)^3}{6(c + 1)^2} \times \dfrac{3(c + 1)^2}{2(c - 1)^2} \div \dfrac{7c}{4(c + 1)}$.
15. $\dfrac{r^2 - 9r + 18}{r^3 - 27} \div \dfrac{r^2 - 3r - 18}{r^2 - 9}$.
16. $\dfrac{x^2 - 7x + 12}{x^2 - 3x - 4} \div \dfrac{x^2 - 5x + 6}{x^2 - x - 2}$.
17. $(x^2 + 2x - 15) \div \dfrac{x + 3}{x^2 - 5x + 25}$.

18. $\dfrac{(a-1)^2-b^2}{(a-b)^2-1} \div \dfrac{a^2-(1-b)^2}{a^2-(b-1)^2}$.

19. $\dfrac{x^2-3x}{x^2+3x} \times \dfrac{x^2+3x+9}{x+3} \div \dfrac{x^3-27}{(x+3)^2}$.

20. $\left(\dfrac{a-b}{a+b}+1\right) \div \dfrac{4ab}{a^2-b^2}$.

21. $\dfrac{2d}{c+2-d}+1 \div \dfrac{(c+d+2)^2}{(c+2)^2-d^2}$.

22. $\dfrac{1}{3}\left(\dfrac{1}{a+2}\right) \div \dfrac{1}{6(a+3)}$.

81. Reciprocals.—Two numbers are *reciprocals* of each other, if their product equals 1.

Thus 4 and $\frac{1}{4}$ are reciprocals, since $4 \times \frac{1}{4} = 1$.

Also if $a \times b = 1$, a is the reciprocal of b.

Since 1 is the product of the two numbers, if 1 is divided by one of the numbers, the quotient will be the reciprocal of that number.

In finding the reciprocal of a fraction, as $\frac{x}{y}$, 1 must be divided by $\frac{x}{y}$. But $1 \div \frac{x}{y} = 1 \times \frac{y}{x} = \frac{y}{x}$. Hence the reciprocal of any fraction is always equal to the fraction inverted.

COMPLEX FRACTIONS

82. A *complex fraction* is a fraction that has one or more fractions in numerator or denominator, or in both. Since a fraction is an indicated division, a complex fraction indicates a division in which dividend and divisor may be expressions that are not integral.

Thus, $\dfrac{x+\frac{4}{x}}{\frac{x^4-16}{xy}}$ indicates that the numerator $x+\frac{4}{x}$ is to be divided by the denominator $\frac{x^4-16}{xy}$.

To perform such a division, numerator and denominator must be changed to fractions.

The mixed expression $x + \frac{4}{x}$ equals $\frac{x^2+4}{x}$ which can be divided by the denominator $\frac{x^4-16}{xy}$ in the usual way.

$$\frac{x^2+4}{x} \div \frac{x^4-16}{xy} = \frac{x^2+4}{x} \times \frac{xy}{(x^2+4)(x^2-4)} = \frac{y}{x^2-4}.$$

Hence, **To Simplify a Complex Fraction,**

Perform the operations indicated in numerator and denominator and divide the simplified numerator by the simplified denominator.

1. Simplify
$$\frac{\frac{1}{a}+\frac{1}{b}}{\frac{a}{b}+\frac{b}{a}}$$

Performing the additions indicated in numerator and denominator,

$$\frac{\frac{a+b}{ab}}{\frac{a^2+b^2}{ab}}.$$

Dividing numerator by denominator,

$$\frac{a+b}{ab} \div \frac{a^2+b^2}{ab} = \frac{a+b}{ab} \times \frac{ab}{a^2+b^2} = \frac{a+b}{a^2+b^2}.$$

2. Simplify $$\frac{1}{1+\frac{1}{1+\frac{1}{1-a}}}$$

As the denominator of this complex fraction contains a complex fraction, the latter should be simplified by the same rule. Changing the mixed expression $1 + \frac{1}{1-a}$ to a fraction,

$$\frac{1}{1+\frac{1}{1+\frac{1}{1-a}}} = \frac{1}{1+\frac{1}{\frac{2-a}{1-a}}} = \frac{1}{1+1\div\frac{2-a}{1-a}} = \frac{1}{1+\frac{1-a}{2-a}}.$$

Changing the mixed expression $1 + \frac{1-a}{2-a}$ to a fraction,

$$\frac{1}{1+\frac{1-a}{2-a}} = \frac{1}{\frac{3-2a}{2-a}} = 1 \div \frac{3-2a}{2-a} = \frac{2-a}{3-2a}$$

The student should note that *simplify* means *to perform the indicated operations.*

If the numerator or denominator of a complex fraction contains no complex fractions, a shorter method of simplifying the fraction is to

Multiply numerator and denominator by the L. C. D. of all the fractions occurring in both terms of the complex fraction.

3. Simplify, $\dfrac{\dfrac{1}{1-x}+\dfrac{1}{1+x}}{\dfrac{x}{1-x}-\dfrac{1}{1+x}}$.

The L. C. D. is $(1-x)(1+x)$.

Multiplying both terms of the complex fraction by the L. C. D.,

$$\frac{1+x+1-x}{x(1+x)-(1-x)}=\frac{2}{x+x^2-1+x}=\frac{2}{x^2+2x-1}.$$

EXERCISE 54

Simplify:

1. $\dfrac{\dfrac{a}{c}+\dfrac{b}{c}}{\dfrac{d}{c}}$.

2. $\dfrac{x+\dfrac{y}{z}}{y-\dfrac{x}{z}}$.

3. $\dfrac{\dfrac{a}{b}-c}{a-\dfrac{b}{c}}$.

4. $\dfrac{\dfrac{x}{a}-\dfrac{a}{x}}{\dfrac{x+a}{a^2x^2}}$.

5. $\dfrac{x+1-\dfrac{x-3}{x-2}}{\dfrac{x-1}{x-2}}$.

6. $\dfrac{\dfrac{a}{3}-\dfrac{3}{a}}{\dfrac{a^2-6a+9}{3a}}$.

7. $\dfrac{\dfrac{x}{x+2}+\dfrac{x}{x-2}}{\dfrac{x}{x-2}-\dfrac{x}{x+2}}$.

8. $\dfrac{\dfrac{a^2+b^2}{c}-a}{\dfrac{1}{b}-\dfrac{1}{a}}$.

9. $\dfrac{m-2-\dfrac{2}{m-5}}{m-3-\dfrac{1}{m-5}}$.

10. $1+\dfrac{1}{1-\dfrac{2}{1+\dfrac{2}{1-x}}}$.

11. $\dfrac{\dfrac{x^2-y^2-1}{2y}-1}{1+\dfrac{y^2-x^2+1}{2y}}.$

12. $\dfrac{\dfrac{a-1}{a+1}+\dfrac{a}{2}-\dfrac{a+1}{a-1}}{\dfrac{a^4-6a^2+5}{a^2(a^2-2)+1}}.$

13. $\dfrac{\dfrac{x^2-9}{x^2+9}+\dfrac{x^2+9}{x^2-9}}{\dfrac{x-3}{x+3}+\dfrac{x+3}{x-3}}.$

14. $\dfrac{a+b}{a+b+\dfrac{1}{a-b+\dfrac{1}{a+b}}}.$

15. $x^2-1-\dfrac{x-1}{1-\dfrac{x}{x+1}}.$

16. $\dfrac{x}{\dfrac{x}{1+x+x^2}+1-x}+1.$

17. $\dfrac{1+\dfrac{2}{a}}{1+\dfrac{1}{2a}}\div\dfrac{a-\dfrac{a}{2}}{2+\dfrac{1}{a}}.$

18. $\dfrac{1}{2a+b-\dfrac{2ab}{2a+b}}-(2a+b).$

19. $\dfrac{\dfrac{x+2}{1-2x}-2}{1+\dfrac{2x+4}{1-2x}}\div\dfrac{1-\dfrac{2x-x^2}{1+2x}}{x+\dfrac{2-x}{1+2x}}.$

20. $\dfrac{\left[1-\dfrac{a^3}{(b+2)^3}\right]\left[\dfrac{a}{b+2}+1\right]}{\left[1-\dfrac{a^2}{(b+2)^2}\right]\left[\dfrac{a^2}{(b+2)^2}+\dfrac{a}{b+2}+1\right]}.$

REVIEW EXERCISE IX

1. Simplify $\dfrac{(x^2-y^2)(2x^2-2xy)}{4(x-y)^2\div\dfrac{xy}{x+y}}.$

2. Find the sum of $\dfrac{2}{x}+\dfrac{3}{1-2x}-\dfrac{2x-3}{4x^2-1}+\dfrac{1}{1+2x}.$

3. Find the product of

$$\frac{a^4-x^4}{a^3+x^3}\times\frac{a+x}{a^2-x^2}\times\frac{a^2-ax+x^2}{(a+x)^2}.$$

4. Simplify $\frac{x^2+x-2}{x^2-5x+4} \div \frac{x^2-x-6}{x^2+x-20}$.

5. Find the sum of

$$\frac{b+c}{(a-b)(a-c)} + \frac{a+c}{(b-c)(b-a)} + \frac{a+b}{(c-a)(c-b)}.$$

6. Multiply

$$\left(1-\frac{y^4}{x^4}\right)\left(\frac{xy}{x^2+y^2}\right)\left(\frac{x^3}{(x-y)^2}\right)\left(\frac{x^3-y^3}{x^2-4xy-5y^2}\right).$$

7. Simplify $\frac{p+2}{x-3} \div \frac{2p^2-8}{3x^2-27} \times \frac{4}{px+3p}$.

8. Simplify $\frac{2-6x}{5} + \frac{4x-1}{2} - \frac{5x-3}{6} - \frac{1-x}{3}$.

9. Reduce to lowest terms: $\frac{3a^2-6a}{a^2+4a+4}$.

10. Find the sum of $\frac{3x-1}{4} - \frac{2x+1}{3} - \frac{4x-5}{5}$.

11. Find the result of $\frac{5c^3}{2ab} \times \frac{3b^3}{5ac} \div \frac{3bc}{2a^3}$.

12. Simplify $\frac{x^2+2x-3}{x^2-2x-3} \div \frac{x^2+4x+3}{x^2-4x+3} \times \frac{x^3+1}{x^3-1}$.

13. Simplify $\frac{a-2}{a^2-a+1} - \frac{1}{a+1} + \frac{a^2+a+3}{a^3+1}$.

14. Simplify

$$\frac{x^2-y^2}{x^2-3xy+2y^2} \times \frac{xy-2y^2}{x^2+xy} + \frac{x^2-xy}{(x-y)^2} \div \frac{y^3}{x^3-y^3}.$$

15. Simplify $\left(1-\frac{x-1}{x^2+6x+5}\right)\left(1-\frac{2}{x^2+7x+12}\right)$

16. Simplify $\frac{x}{2y-1} + \frac{y}{2y+1} - \frac{y-x}{1-4y^2}$.

17. Simplify $\frac{5x+2}{x^2-4} + \frac{2}{x-2} + \frac{3}{x+2}$.

18. Add $\frac{1}{a+b} - \frac{1}{a-b} - \frac{2a}{b^2-a^2}$.

19. Simplify $\frac{x^3+y^3}{a^3+b^3} \times \frac{a^4+a^2b^2+b^4}{a^3-b^3} \times \frac{x^3-y^3}{x^4-2x^2y^2+y^4}$.

20. Simplify $\frac{x^3-27}{4x^2-9} \times \frac{4x^2+12x+9}{8x^2+12x+18} \div \frac{6x^2+5x-6}{4x-6}$.

21. Simplify $\dfrac{a+1}{a^2-a-6}-\dfrac{a-4}{a^2-4a+3}+\dfrac{a+3}{a^2+a-2}$.

22. Add $\dfrac{3-x}{1-3x}-\dfrac{3+x}{1+3x}+\dfrac{1-16x}{1-9x^2}$.

23. Simplify

$$\frac{x^2+x}{x^2-x}\times\frac{3x^2-3}{3x+3}\div\frac{x^2-2x-3}{x^3-9x}\div\frac{x^2+x-6}{6x^2}.$$

24. Simplify $1+\dfrac{1+\dfrac{1}{x-1}}{1-\dfrac{1}{x-1}}$.

25. Simplify $\dfrac{x+y}{y}-\dfrac{2x}{x+y}+\dfrac{x^2y-x^3}{y(x^2-y^2)}$.

26. Simplify $\dfrac{\dfrac{1}{a^2}+\dfrac{1}{b^2}}{\dfrac{1}{a^2}-\dfrac{1}{b^2}}\div\dfrac{\dfrac{1}{a}+\dfrac{1}{b}}{\dfrac{1}{a}-\dfrac{1}{b}}$.

27. Simplify $\dfrac{\dfrac{1}{ab}-\dfrac{1}{ac}-\dfrac{1}{bc}}{\dfrac{a^2-(b-c)^2}{a}}$.

28. Simplify $\dfrac{\dfrac{a^2+b^2}{b}-a}{\dfrac{1}{b}-\dfrac{1}{a}}\times\dfrac{a^2-b^2}{a^3+b^3}$.

29. Simplify $\dfrac{\dfrac{x-\dfrac{15}{x+2}}{x-7}}{1-\dfrac{3}{x+5}}$.

30. Simplify $\dfrac{\dfrac{m^3-n^3}{m^3+n^3}\left(1-\dfrac{2n}{m+n}\right)}{1+\dfrac{2mn}{m^2-mn+n^2}}$.

CHAPTER X

FRACTIONAL EQUATIONS

83. Equations containing fractions are called *Fractional Equations.*

SOLUTION OF FRACTIONAL EQUATIONS

1. Solve $\frac{2x-1}{4} - \frac{4x+5}{3} + \frac{2x+1}{2} = 5.$

Since both sides of any equation may be multiplied by the same number without disturbing their equality, both sides of this equation may be multiplied by 12, the L. C. M. of the denominators. Each denominator being contained in 12, the denominators will be cancelled by this multiplication and the equation cleared of fractions.

Multiplying by 12,

$$3(2x-1) - 4(4x+5) + 6(2x+1) = 60$$

Removing the parentheses,

$$6x - 3 - 16x - 20 + 12x + 6 = 60$$

Transposing,

$$6x - 16x + 12x = 60 + 3 + 20 - 6$$
$$2x = 77$$
$$x = \frac{77}{2}$$

2. Solve $\frac{2x+1}{3x-3} = \frac{7x-1}{6x+6} - \frac{2x^2-3x-45}{4x^2-4}.$

Factoring the denominators,

$$\frac{2x+1}{3(x-1)} = \frac{7x-1}{6(x+1)} - \frac{2x^2-3x-45}{4(x+1)(x-1)}$$

Multiplying the equation by the L. C. M. of the denominators,

$$12(x+1)(x-1),$$

$$4(x+1)(2x+1) = 2(x-1)(7x-1) - 3(2x^2-3x-45)$$

Removing the parentheses,

$$8x^2 + 12x + 4 = 14x^2 - 16x + 2 - 6x^2 + 9x + 135$$

Transposing,

$$8x^2 - 14x^2 + 6x^2 + 12x + 16x - 9x = -4 + 2 + 135$$

EXERCISE 52

Find the product of:

1. $\frac{4y^2}{x^2} \times \frac{4x^2}{3y^2} \times \frac{2x}{3y}.$
2. $\frac{3x^2y^2z^2}{4mnr} \times \frac{10m^2n^2r^3}{15x^3y^3z^3}.$
3. $\frac{2abc}{5x^2y} \times \frac{4abx^3}{10a^2b^2y^3} \times \frac{25a^2bx^2y^2}{32abxy}.$
4. $\frac{2}{x+2} \times \frac{x^2-4}{10}.$
5. $\frac{a-b}{a+b} \times \frac{a^2-b^2}{a-b}.$
6. $(x+y) \times \frac{1}{x^3+y^3}.$
7. $\frac{a^2+2ab+b^2}{a^3-b^3} \times \frac{a-b}{a^2-b^2}.$
8. $(m^2+mn+n^2) \times \frac{m+n}{m^3-n^3}.$
9. $\frac{3a(a+3b)}{a^3+b^3} \times \frac{a^2-ab+b^2}{a^2-9b^2}.$
10. $\frac{m+n}{m^4-n^4} \times \frac{m^2-n^2}{(m+n)^2}.$
11. $\frac{x+y}{x^6-y^6} \times \frac{x^3+y^3}{x^2-y^2}.$
12. $\frac{3a(a-b)}{5b(a+b)} \times \frac{15(a^2-b^2)}{9(a-b)^2}.$
13. $\frac{16c^3b}{7(c^3-b^3)} \times \frac{35(c^2+cb+b^2)}{14c^2b^2}.$
14. $\frac{x^2+3x+2}{x^2-5x+6} \times \frac{x^2-7x+12}{x^2+8x+7}.$
15. $\frac{a^2-a}{a^2-1} \times \frac{2a^2-4a+2}{4a^2+6a}.$
16. $\frac{a^2+a-2}{a^2-3a} \times \frac{a^2+9a-36}{a^3+2a^2}.$
17. $\frac{a^2-6a-16}{a^2+4a-21} \times \frac{a^2-8a+15}{a^2+9a+14}.$
18. $\frac{a^2-1}{a^2-4a-5} \times \frac{a^2-25}{a^2+2a-3}.$
19. $\frac{a^2+9ab+18b^2}{a^2-9ab+20b^2} \times \frac{ab^2-4b^3}{a^2+6ab+9b^2}.$
20. $\frac{m^2+4m-32}{m^2-8m+15} \times \frac{m^2-10m+21}{m^2+m-56}.$
21. $\frac{x^2+2x-15}{x^2-9} \times \frac{x^2-11x+30}{x^3-6x^2+9x} \times \frac{x^2-3x}{x^2-25}.$

22. $\dfrac{y(x-y)}{x^2+2xy+y^2} \times \dfrac{x^2-y^2}{xy^2} \times \dfrac{x^2-y^2}{xy}.$

23. $\dfrac{5a^2+9a-2}{7a^2+17a-12} \times \dfrac{6a^2-11a-10}{3a^2+2a-5} \times \dfrac{21a^2+23a-20}{10a^2-27a+5}.$

24. $\dfrac{7r^4-7r^3}{7r^2-56r-63} \times \dfrac{r^2+r-90}{r^4-9r^3+8r^2}.$

25. $\dfrac{a^2-b^2+2bc-c^2}{x+y+z} \times \dfrac{x^2+y^2+2xy-z^2}{a+b-c}.$

26. $\dfrac{3x^2-5x+2}{4x^2-5x+1} \times \dfrac{4x^2-17x+4}{6x^2-7x+2} \times \dfrac{10x^2-21x+9}{5x^2-23x+12}.$

27. $\left(x-\dfrac{y^2}{x}\right)\left(y+\dfrac{x^2-y^2}{y}\right).$

28. $\dfrac{ac+bc-ad-bd}{xy-4x-3y+12} \times (xy-4x+3y-12).$

29. $\dfrac{(a+b)^2-c^2}{a^2+ab-ac} \times \dfrac{a}{(a-c)^2-b^2} \times \dfrac{(a-b)^2-c^2}{ab-b^2+bc}.$

30. $\dfrac{6t^2-7t+2}{10t^2-7t+1} \times \dfrac{6t^2-5t+1}{6t^2+t-1} \times \dfrac{10t^2+3t-1}{5t^2-4t-1}.$

DIVISION OF FRACTIONS

80. Fractions are divided by the same rule as in arithmetic.

Thus, $\dfrac{5}{7} \div \dfrac{15}{14} = \dfrac{5}{7} \times \dfrac{14}{15} = \dfrac{2}{3}.$

In the same way, $\dfrac{a}{b} \div \dfrac{c}{d} = \dfrac{a}{b} \times \dfrac{d}{c} = \dfrac{ad}{bc}.$

Hence, To Divide by a Fraction,

Invert the divisor and proceed as in multiplication.

As in multiplication of fractions, all mixed expressions or integral expressions are changed to fractions.

1. Find the quotient of,

$$\frac{x^2}{y^2} \div \frac{x}{y} = \frac{x^2}{y^2} \times \frac{y}{x} = \frac{x}{y}.$$

As the remaining fraction is complex, simplify it by multiplying numerator and denominator by 3, which will not change the value of the fraction,

$$13 = -\frac{108x + 18}{3x - 1}.$$

Clear of fractions by multiplying the equation by $3x - 1$,

$$13(3x - 1) = -(108x + 18)$$

Simplifying, $$39x - 13 = -108x - 18$$

Transpose and collect like terms,

$$147x = -5$$

$$\therefore \quad x = -\tfrac{5}{147}$$

EXERCISE 56

Solve:

1. $\dfrac{1}{x+1} - \dfrac{1}{x+2} = \dfrac{1}{x+3} - \dfrac{1}{x+4}.$

2. $\dfrac{7x+3}{5} - \dfrac{21x+9}{15} = \dfrac{17x-3}{3x+11} + 2.$

3. $\dfrac{1}{x-1} - \dfrac{2}{2x+1} = \dfrac{1}{x-2} - \dfrac{2}{2x-1}.$

4. $\dfrac{11y-15}{10} - \dfrac{33y+15}{30} = \dfrac{5y+5}{y-5}.$

5. $\dfrac{2y+8\frac{1}{2}}{9} - \dfrac{13y-2}{17y-32} + \dfrac{y}{3} = \dfrac{7y}{12} - \dfrac{y+16}{36}.$

6. $\dfrac{4-9x}{12} + \dfrac{3x-4}{3} - \dfrac{x+11}{12x-27} - \dfrac{x}{4} = 0.$

7. $\dfrac{y+2}{y+5} - \dfrac{y+3}{y+6} = \dfrac{y+4}{y+7} - \dfrac{y+5}{y+8}.$

8. $\dfrac{t-5}{t-7} - \dfrac{t-4}{t-6} = \dfrac{t-8}{t-10} - \dfrac{t-7}{t-9}.$

9. $\dfrac{2x+1}{3} - \dfrac{4x-1}{6} = \dfrac{75}{6} - \dfrac{3x+5}{2x-5}.$

10. $\dfrac{4z-3}{10} - \dfrac{2z-5}{5} = \dfrac{z-3}{2z-15} + 1\tfrac{1}{10}.$

11. $\dfrac{x-5}{x-6} - \dfrac{x-4}{x-5} = \dfrac{x-8}{x-9} - \dfrac{x-7}{x-8}.$

LITERAL EQUATIONS

85. Equations, in which some or all of the known numbers are expressed by letters, are called *literal equations*. It is usual to express known numbers by letters near the beginning of the alphabet.

Literal equations are solved by the same methods as numerical equations.

1. Solve $3a - 4cx = 7ab - 6bx$.

Transpose the terms containing the unknown number x to the left and the known terms to the right,

$$6bx - 4cx = 7ab - 3a$$

Collect the coefficients of x,

$$(6b - 4c)x = 7ab - 3a$$

Divide by $6b - 4c$, the coefficient of x,

$$x = \frac{7ab - 3a}{6b - 4c}$$

Note that collecting the coefficients of x is similar to collecting like terms. Thus, $6x - 4x = (6 - 4)x = 2x$. In this case the like terms are added by adding their coefficients and, as these are numbers, they can actually be combined. If the coefficients are literal, this addition can only be indicated by connecting them by their proper signs and enclosing them in a parenthesis.

2. Solve $\dfrac{1}{x - 2c} - \dfrac{1}{x - 6c} = \dfrac{1}{x - 4c} - \dfrac{1}{x - 8c}$.

Add the fractions on each side. The L. C. M. on the left side is $(x - 2c)(x - 6c)$ and on the right side $(x - 4c)(x - 8c)$

$$\frac{(x - 6c) - (x - 2c)}{(x - 2c)(x - 6c)} = \frac{(x - 8c) - (x - 4c)}{(x - 4c)(x - 8c)}$$

Remove parentheses, $\dfrac{x - 6c - x + 2c}{(x - 2c)(x - 6c)} = \dfrac{x - 8c - x + 4c}{(x - 4c)(x - 8c)}$.

Collect like terms, $\dfrac{-4c}{(x - 2c)(x - 6c)} = \dfrac{-4c}{(x - 4c)(x - 8c)}$.

As these equal fractions have equal numerators, their denominators must also be equal,

$$(x - 2c)(x - 6c) = (x - 4c)(x - 8c)$$

Multiply, $x^2 - 8cx + 12c^2 = x^2 - 12cx + 32c^2$

Transpose, $x^2 - x^2 - 8cx + 12cx = 32c^2 - 12c^2$

Collect like terms, $4cx = 20c^2$

Divide by $4c$, the coeff. of x, $x = \dfrac{20c^2}{4c} = 5c$

If the equations contain complex fractions, these should be simplified before proceeding with the solution

3. Solve $\dfrac{\dfrac{a+b}{x}}{\dfrac{1}{a}} = \dfrac{m}{n}.$

Simplify,

$$\frac{a+b}{x} \div \frac{1}{a} = \frac{m}{n}$$

$$\frac{a+b}{x} \times \frac{a}{1} = \frac{m}{n}$$

$$\frac{a^2+ab}{x} = \frac{m}{n}.$$

Clear of fractions, $n(a^2 + ab) = mx$

Transpose and change both signs,

$$mx = n(a^2 + ab)$$

Divide by m, $\quad x = \dfrac{n(a^2 + ab)}{m}$

EXERCISE 57

Solve:

1. $bx - 4b = 3ax - 2a.$
2. $(a + b)x - 4a^2 = (a - b)x.$
3. $c(x + d) + d(x + c) = 1 + 2cd.$
4. $(x + m)(x + 1) = (x + m + 1)^2.$
5. $(a - bx)(b - ax) = ab(x^2 - 1).$
6. $\dfrac{1}{a} - \dfrac{1}{y} = \dfrac{1}{y} - \dfrac{1}{b}.$
7. $\dfrac{ab}{x} - cd = \dfrac{1}{x} - e.$
8. $\dfrac{x+a}{x+b} = \dfrac{x+b}{x+a}.$
9. $\dfrac{x}{a - ax} = \dfrac{c}{b - bx}.$
10. $\dfrac{2x - a}{x - a} - \dfrac{x - a}{x + a} = 1.$
11. $5x + 15\left(1 - \dfrac{a}{3}\right) = a(x - a).$
12. $\dfrac{a + b}{3 + x} = \dfrac{a - b}{3 - x}.$
13. $\dfrac{x}{m} + \dfrac{x}{n} + \dfrac{x}{p} = q.$
14. $\dfrac{3ax}{a - b} - 5a = 8x.$
15. $\dfrac{x}{m - n} - \dfrac{5m}{m + n} = \dfrac{2nx}{m^2 - n^2}.$

16. $\frac{1}{a} - \frac{x-a}{ax} = \frac{a}{x+a}.$

17. $\frac{y+2-a}{y-2-b} = \frac{y+1-a}{y+1-b}.$

18. $\frac{z-m}{3} = \frac{(z-n)^2}{3z-m}.$

19. $\dfrac{\dfrac{a}{a-2}}{\dfrac{1}{x+1}} = \dfrac{\dfrac{a}{a+2}}{\dfrac{1}{x-1}}.$

20. $\dfrac{a - \dfrac{x}{a-1}}{a + \dfrac{x}{a-1}} = a + 1.$

21. $\frac{a+x}{a} - \frac{2x}{a+x} - \frac{x^2(x-a)}{a(x^2-a^2)} = \frac{1}{3}.$

22. $\frac{bx}{b+x} + \frac{cx}{c+x} - (b+c) = 0.$

23. $\frac{cx}{b} - c = \frac{b-x}{2a} - \frac{c(b-x)}{3d}$

24. $\frac{1}{x-2b} - \frac{1}{x-3b} = \frac{1}{x-4b} - \frac{1}{x-5b}.$

25. $\frac{x-a}{x-3a} - \frac{x+2a}{x-4a} = \frac{5ax+9a^2}{x^2-7ax+12a^2}.$

PROBLEMS INVOLVING FRACTIONAL EQUATIONS

86. In solving problems, the student should note that the letter x which is commonly used, represents the *number* of units contained in the unknown quantity and not the unknown quantity itself. The equation is formed by finding two equal numbers, one or both containing the letter x. Thus, we may form an equation stating that the number of dollars in the price of a lot of ground is equal to the number of square feet the lot contains, but it would be absurd to say that the price of the lot is equal to the area of the lot.

EXERCISE 58

1. The sum of the third and sixth parts of a number diminished by 20 equals the difference between the sixth and seventh parts of the number. Find the number.

Let x = the unknown number

Then $\frac{x}{3} + \frac{x}{6}$ = the sum of the third and sixth parts

And $\frac{x}{6} - \frac{x}{7}$ = the difference between the sixth and seventh parts

Then $\frac{x}{3} + \frac{x}{6} - 20 = \frac{x}{6} - \frac{x}{7}$.

Multiply the equation by 42, the L. C. M. of the denominators,

$$14x + 7x - 840 = 7x - 6x$$

Transposing and combining like terms,

$$20x = 840$$

Dividing by 20, $x = 42$

2. Find the number whose third part exceeds the fifth part by 5.

3. One-third of a certain number exceeds the sum of its ninth and eleventh parts by 13. Find the number.

4. Three times the difference of the third and fourth part of a number is 7 less than the sum of its fifth and sixth parts. Find the number.

5. Find the three consecutive numbers, x, $x + 1$, $x + 2$, so that $\frac{1}{6}$ the first plus $\frac{1}{5}$ the second will equal $\frac{1}{3}$ the third number.

6. The sum of two numbers is 73. If the greater is divided by the smaller number, the quotient is 2 and the remainder 1. Find the numbers.

Let x = the greater number

Then $73 - x$ = the smaller number

Since the quotient $= \frac{\text{Dividend} - \text{Remainder}}{\text{Divisor}}$

Therefore, $\frac{x - 1}{73 - x} = 2$

Clearing of fractions, $x - 1 = 146 - 2x$

Transposing and combining like terms,

$$3x = 147$$

Dividing by 3, $x = 49$ the larger number

$73 - x = 24$ the smaller number.

7. The sum of two numbers is 125 and if the greater is divided by the smaller number, the quotient is 2 and the remainder 11. Find the numbers.

8. Divide 61 into two parts so that if the greater is divided by the smaller, the quotient is 4 and the remainder 6.

9. The difference of two numbers is 18 and if the larger is divided by the smaller, the quotient is 2 and the remainder 5. Find the numbers.

10. If the sum of three consecutive numbers is divided by the smallest numbers the quotient is 3 and the remainder 3. Find the numbers.

11. A son is one-fourth as old as his father. Four years ago he was only one-sixth as old as his father. Find the age of each.

Let x = the number of years in the father's age

Then $\frac{x}{4}$ = the number of years in the son's age

Four years ago,

$x - 4$ = the number of years in father's age

$\frac{x}{4} - 4$ = the number of years in son's age

Therefore $\frac{x}{4} - 4 = \frac{1}{6}(x - 4)$

Clearing of fractions by multiplying the equation by 12,

$$3x - 48 = 2(x - 4)$$
$$3x - 48 = 2x - 8$$

Transposing and combining,

$$x = 40$$

and $$\frac{x}{4} = 10$$

Therefore the father is 40 years old and the son 10 years.

12. A son is one-fifth as old as his father. Two years ago he was only one-sixth as old as his father. Find the age of each.

13. A father is three times as old as his son. In twelve

Clearing of fractions,

$$\begin{aligned} 3x + 2x - x &= 30 \\ 4x &= 30 \\ x &= 7\tfrac{1}{2} \end{aligned}$$

∴ The cistern will be filled in $7\frac{1}{2}$ hours.

24. A can do a piece of work in 2 days and B in 3 days. In how many days can both together do the work?

25. A cistern can be filled by three pipes in 10 hours, 15 hours and 20 hours respectively. In how many hours will it be filled if all the pipes are running together?

26. A can do a piece of work in $3\frac{1}{3}$ days, B in $4\frac{1}{4}$ days, and C in $5\frac{1}{5}$ days. In how many days could they all together do the work?

27. A cistern can be filled by two pipes in 1 hour 45 minutes, and 2 hours 30 minutes respectively and emptied by a third pipe in 4 hours 15 minutes. In how many hours will the cistern be filled if all the pipes are open?

28. A and B together can mow a field in 10 hours. B and C together can mow it in 12 hours. A and C together can mow it in 11 hours. In how many hours could they all together mow the field?

29. A tank has three pipes. All three pipes together can fill the tank in 4 hours. The first pipe alone would fill it in 8 hours and the second in 10 hours. How long would it take the third pipe alone?

30. A and B together can dig a ditch in 9 hours, A and C together can do it in 10 hours. C alone can do it in 15 hours. In how many hours can B and C together do it?

31. A courier who travels at the rate of 5 miles per hour, is followed, after 3 hours, by a second courier who travels $6\frac{1}{2}$ miles per hour. In how many hours will the second courier overtake the first?

Let x = the number of hours the second courier travels
Then $x + 3$ = the number of hours the first courier travels
$5(x + 3)$ = the number of miles the first courier travels
$6\frac{1}{2}x$ = the number of miles the second courier travels

If the second courier overtakes the first, he must travel the same distance.

Therefore,

$$6\tfrac{1}{2}x = 5(x + 3)$$
$$\tfrac{13}{2}x = 5x + 15$$
$$13x = 10x + 30$$
$$3x = 30$$
$$x = 10$$

Therefore the second courier will overtake the first in 10 hours.

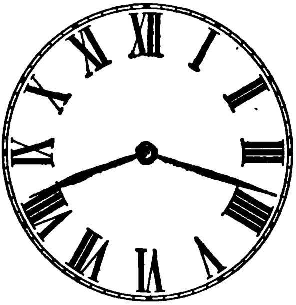
32. A courier, who travels at the rate of 7 miles per hour, is followed, after 2 hours, by a second courier who travels 7½ miles an hour. In how many hours will the second courier overtake the first?

33. An automobilist who travels at 20 miles per hour, starts from A toward B, 150 miles distant, and 2 hours later a second automobilist, who travels at 35 miles per hour, starts from B toward A. In how many hours will they meet and how far from A is the point of meeting?

34. An aviator flies, with the wind, at 70 miles per hour and returns, against the wind, at 60 miles per hour. If the entire trip took 13 hours, how far from his base did he fly?

35. In going a certain distance, a train traveling at the rate of 40 miles per hour, takes 2 hours less time than a train traveling at the rate of 30 miles per hour. Find the distance.

36. Find the time between 3 and 4 o'clock, when the hands of a clock point in opposite directions.



Since, in one complete revolution the minute hand travels over 60 minute spaces, the minute hand is exactly 30 minute spaces from the hour hand when the two hands point in opposite directions.

Let x = the number of minute spaces the hour hand has moved over since 3 o'clock.
Then $x + 45$ = the number of minute spaces the minute hand has moved over since 3 o'clock, as the minute hand must lie between 9 and 10.

In one hour, the minute hand moves over 60 minute spaces while the hour hand moves over 5 spaces. Hence the minute hand moves 12 times as fast as the hour hand.

Therefore
$$12x = x + 45$$
$$11x = 45$$
$$x = 4\tfrac{1}{11}$$

The minute hand is therefore $4\frac{1}{11}$ minute spaces beyond the 9. Hence the time is $49\frac{1}{11}$ minutes after 3 o'clock.

37. Find the time between 5 and 6 o'clock when the hands of a clock are together.

38. Find the time between 2 and 3 o'clock when the hands of a clock point in opposite directions.

39. Find the time between 1 and 2 o'clock, when the hands of a clock are at right angles.

40. A hound takes 3 leaps while a hare takes 4; but 2 of the hound's leaps are equivalent to 3 of the hare's. The hare has a start of 100 of her own leaps. How many leaps must the hound take to catch the hare?

Let $3x$ = the number of leaps the hound takes
Then $4x$ = the number of leaps the hare takes
Let a = the number of feet in one leap of the hare
Then $\frac{3a}{2}$ = the number of feet in one leap of the hound.
Hence $3x \times \frac{3a}{2}$ = the number of feet in the whole distance

As the hare has a start of 100 leaps,
$a(100 + 4x)$ = the number of feet in the whole distance

Therefore $\frac{9ax}{2} = a(100 + 4x)$
Multiplied by 2, $9ax = 2a(100 + 4x)$
Divided by a, $9x = 2(100 + 4x)$

Then $9x = 200 + 8x$

$x = 200$

and $3x = 600$

Hence the hound must take 600 leaps to catch the hare.

41. A rabbit takes 5 leaps while a hound takes 3; but 1 of the hound's leaps is equivalent to 2 of the rabbit's. The rabbit has a start of 200 of her own leaps. How many leaps must the hound take to catch the rabbit?

42. A dog takes 5 leaps while a hare takes 7; but 5 of the dog's leaps are equivalent to 9 of the hare's. The hare has a start of 100 of her own leaps. How many leaps will the hare take before she is caught?

43. A rabbit takes 6 leaps while a dog takes 5; but 2 of the dog's leaps are equivalent to 3 of the rabbit's. If the rabbit has a start of 100 of the dog's leaps, how many leaps must the dog take to catch the rabbit?

44. A merchant adds yearly to his capital ½ of it, but takes from it, at the end of each year, $1500 for expenses. At the end of the third year, after deducting the last $1500, he has 2⅝ of his original capital. Find his original capital.

Let x = the number of dollars in his original capital.

Then $\frac{3x}{2} - 1500 = \frac{3x - 3000}{2}$ = the number of dollars he had at the end of the first year.

This becomes his capital for the second year.

Therefore $\frac{3}{2}\left(\frac{3x - 3000}{2}\right) - 1500 = \frac{9x - 9000}{4} - 1500 = \frac{9x - 15,000}{4}$ = the number of dollars he had at the end of the second year.

And $\frac{3}{2}\left(\frac{9x - 15000}{4}\right) - 1500 = \frac{27x - 45000}{8} - 1500 = \frac{27x - 57,000}{8}$ = the number of dollars he had at the end of the third year.

Hence $\frac{27x - 57,000}{8} = 2\frac{5}{8}x = \frac{21}{8}x$

Multiplied by 8, $27x - 57,000 = 21x$

$6x = 57,000$

$x = 9500$

Therefore his original capital was $9500.

45. A merchant adds yearly to his capital $\frac{1}{3}$ of it but spends $1000 each year for sundry expenses. At the end of the third year his original capital was doubled. What was his original capital?

46. A trader adds yearly to his capital $\frac{1}{5}$ of it, but takes from it, at the end of each year $5000 for expenses. At the end of the third year, after deducting the last $5000, he has $1\frac{7}{10}$ of his original capital. Find his original apital.

47. A man bequeathed $\frac{3}{5}$ of his estate to his relatives, $\frac{1}{15}$ to a church, $\frac{1}{10}$ to a library and the balance, $16,000, to a hospital. What was the total value of his estate?

48. Find a fraction whose value is $\frac{5}{8}$ and whose denominator is 15 greater than its numerator.

49. The sum of the ages of a father and son is 85 years. If each were 5 years older, the son's age would be $\frac{7}{12}$ of his father's age. Find the age of each.

50. The length of a rectangle is 7 ft. more than its width. If the length were decreased 3 ft. and the width increased 2 ft. the area would be the same. Find its dimensions.

51. A and B together can load a car in 6 hours, B and C in 5 hours, and A and C in $4\frac{2}{7}$ hours. How long would it take each alone to load it?

52. The distance between Southampton and New York is 3068 nautical miles or knots. A vessel leaves New York for Southampton and sailed at the rate of 20 knots an hour. 18 hours later, a second vessel sails from Southampton for New York at the rate of 22 knots per hour. How far from New York will the vessels meet?

53. At what time, between 10 and 11 o'clock will the hands of a clock point in opposite directions?

54. A man has 10 hours at his disposal. How far can he walk at the rate of 4 miles per hour so as to return in time, riding by train at the rate of 36 miles per hour?

55. If silver loses $\frac{1}{10}$ of its weight and gold $\frac{1}{19}$ of its weight, when weighed in water, how many ounces of silver and of

gold in a mass of silver and gold that weighs 476 ounces in air and 432 ounces when weighed in water?

56. The front and rear wheels of a wagon are $9\frac{1}{2}$ ft and 11 ft. respectively in circumference. How far will the wagon have travelled when the front wheel has made 200 revolutions more than the rear wheel?

57. A man paid a bill of $75 out of a sum he had, then deposited $\frac{7}{8}$ of the remainder in bank and had $50 left. How much had he at first?

FORMULAS

87. A *formula* is a literal equation which expresses in algebraic symbols some rule or principle used in the solution of problems in the various sciences or in commercial life.

Thus the rule for finding the interest on a sum of money, at a given rate, for a given number of years may be expressed as a formula.

$$i = prt$$

where i = interest
p = principal
r = rate
t = time.

This formula may be used to compute interest, as follows:

1. Find the interest on $500 at 5% for 3 years and 6 mos.

In this problem, $p = \$500$, $r = 5\%$, $t = 3\frac{1}{2}$ yrs.

Substituting in the formula $i = prt$, remembering that 5% means 5 per hundred or $\frac{5}{100}$,

$$i = \overset{5}{\cancel{500}} \times \frac{5}{\cancel{100}} \times \frac{7}{2} = \frac{175}{2} = \$87.50$$

2. Find the rate if the interest on $900 for 8 months is $36.

Here the unknown quantity is the rate.

Hence, solving the formula $i = prt$ for the unknown quantity,

$$r = \frac{i}{pt}$$

Substituting in the formula,

$$r = \frac{36}{900 \times \frac{2}{3}} = \frac{36}{\frac{1800}{3}} = \frac{36}{600} = \frac{6}{100}$$

$$\therefore r = 6\%$$

Similarly, since the amount is the sum of principal and interest, if a = amount, we have the formula

$$a = p + prt$$

3. In what time will \$1000 amount to \$1250 if put at interest at 5%?

Solving the formula for the unknown quantity t,

$$\begin{aligned} a &= p + prt \\ prt &= a - p \\ t &= \frac{a-p}{pr} \end{aligned}$$

Substituting the values of a, p, and r,

$$t = \frac{1250 - 1000}{1000 \times \frac{5}{100}} = \frac{250}{50} = 5$$

$$\therefore t = 5 \text{ years.}$$

4. Find the interest of \$1250 for 3 years at $5\frac{1}{2}\%$.

5. Find the interest of \$1750 for 2 years 9 months at 5.4%.

6. Find the amount of \$1000 for 6 years 7 months at $4\frac{3}{4}\%$.

7. Find the rate if the interest on \$2000 for 18 months is \$125.

8. Find the time required for the interest on \$2550 at $4\frac{1}{4}\%$ to be \$225.

9. What principle will produce \$165 interest in 2 years 5 months at 6%?

10. In what time will a sum of money double itself at 6% interest?

11. What sum at interest at 5% will produce an income of \$100 per month?

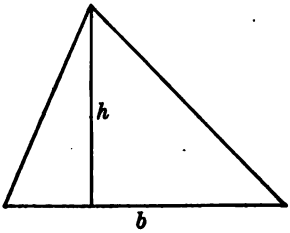
12. At what rate must \$50,000 be invested to produce \$2000 annually?

13. In what time will \$1400 invested at $4\frac{1}{4}\%$ amount to \$2000?

14. The area of a triangle = $\frac{1}{2}$ base × altitude or $A = \frac{1}{2}bh$, where A = area, b = base, h = altitude. Find the altitude of a triangle whose base is 50 ft. and whose area is 625 square feet?

Solving the formula for the unknown quantity h,

$$h = \frac{2A}{b}$$



Substituting the values of A and b,

$$h = \frac{\cancel{2} \times \overset{25}{\cancel{625}}}{\underset{\cancel{25}}{\cancel{50}}} = 25$$

$$\therefore h = 25 \text{ feet.}$$

Note that the letters in the formulas usually represent the initials of the quantities involved.

15. What is the base of a triangle whose area is 200 square inches and whose base is 3 feet 4 inches?

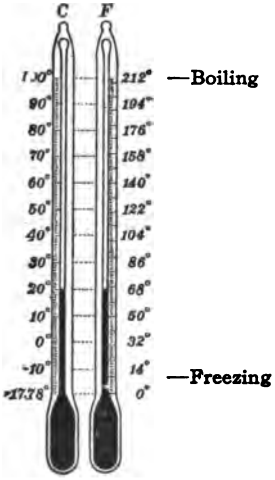
Solve the following formulas for each of the quantities involved:

16. $F = Ma.$
17. $F = \mu W.$
18. $W = Fs.$
19. $Mv_1 = mv_2.$
20. $PV = P_0V_0.$
21. $S = \frac{1}{2}a(2t - 1).$
22. $C = \frac{5}{9}(F - 32).$
23. $WD = pd.$
24. $a = \dfrac{v_1 - v_0}{t}.$
25. $C = \dfrac{1}{C_1} + \dfrac{C}{C_2}.$
26. $R = \dfrac{1}{\dfrac{1}{r_1} + \dfrac{1}{r_2}}.$
27. $I = \dfrac{E}{R}.$
28. $\dfrac{E}{e} = \dfrac{R + r}{r}.$
29. $D = \dfrac{M}{V}.$
30. $S = \dfrac{F}{A}.$
31. $\dfrac{C}{100} = \dfrac{F - 32}{180}.$
32. $H = ms(t_2 - t_1).$
33. $\dfrac{1}{F} = \dfrac{1}{f_1} + \dfrac{1}{f_2}.$
34. $\dfrac{pv}{T} = \dfrac{p'v'}{T'}.$

35. What is the total pressure (P) on the bottom of a swimming tank 50 feet square, filled with water to a depth (h) of 10 feet? Use the formula $P = Ahd$, where A = area of surface and d = weight of a unit of volume of the liquid. Water weighs 62.4 pounds per cubic foot.

36. Using the above formula, find the pressure per square foot at a depth of 3 miles in the ocean, sea water weighing 64 pounds per cubic foot.

37. If the acceleration (change of velocity per unit of time) of a marble rolling down an inclined plane is 50 centimeters per second per second, find its velocity at the end of 3 seconds from rest. Use the formula $v = at$.



38. Find the distance (s) the marble has covered, in problem 37, at the end of 3 seconds from rest, when $s = \frac{1}{2}at^2$.

39. A stone dropped from a bridge strikes the water 3 seconds after leaving the hand. Find the height of the bridge above the water, if the acceleration of gravity is 32.16 feet per second per second.

40. A correct Fahrenheit thermometer reading (F) of the temperature of a room is 70°. An incorrect Centigrade thermometer reading (C) in the same room is 20°. Find the error in the latter if $\frac{F - 32}{180} = \frac{C}{100}$.

41. The formula $\frac{pv}{T} = \frac{p'v'}{T'}$ shows the relations of pressure, volume and temperature of a gas.

If 1000 cubic c.m. of air at 20° C. is changed in temperature to 50°C., what will be its volume, if the pressure remains constant?

42. The amount of heat (H) in calories generated by an electric current of I amperes flowing through a wire of R ohms resistance, for t seconds is $H = 0.24 \times I^2RT$. Find the number of calories of heat generated in a quarter of an hour by a current of 10 amperes passing through a resistance of 2 ohms.

REVIEW EXERCISE X

1. Find the value of x in $\frac{x^2 - a}{bx} - \frac{a - x}{b} = \frac{2x}{b} - \frac{a}{x}$.

2. Find x: $\frac{x^2 - x + 1}{x - 1} + \frac{x^2 + x + 1}{x + 1} = 2x$.

3. Solve for x: $\frac{2x - 3}{2x - 4} - 6 = \frac{x + 5}{3x - 6} - \frac{11}{2}$.

4. Solve: $\frac{2x + 1}{2x - 1} - \frac{10}{4x^2 - 1} = \frac{2x - 1}{2x + 1}$.

5. Solve: $\frac{2x + 19}{5x^2 - 5} - \frac{17}{x^2 - 1} - \frac{3}{1 - x} = 0$.

6. Find value of x:
$$\frac{7x + 9}{8} - \frac{3x + 1}{7} = \frac{9x - 13}{4} - \frac{249 - 9x}{14}.$$

7. Solve: $\frac{7z + 2}{6} - \frac{12 - z}{4} + \frac{z + 2}{2} = 6$.

8. Solve for p: $\frac{6p + 1}{15} - \frac{2p - 4}{7p - 13} = \frac{2p - 1}{5}$.

9. Solve for x: $\frac{4x}{3} = \frac{21x - 25}{12} - \frac{4(3x - 2)}{2}$.

10. Solve for x: $\frac{a}{x} = \frac{x - a^2}{bx} - \frac{1}{a}$.

11. An automobile runs 10 miles an hour faster than a bicycle, and it takes the bicycle 10 hours longer to run 144 miles than the automobile. Find the rate of each.

12. Solve for x: $\frac{3x - 1}{2x - 1} - \frac{4x - 2}{3x - 1} = \frac{1}{6}$.

13. Solve for x: $\frac{x + 1}{2} - 3\frac{1}{3} = \frac{2 - 2x}{6} + \frac{x + 4}{3}$.

14. Solve for x: $\frac{m^2x + n}{mx} - \frac{n^2x + m}{nx} = \frac{m - n}{mnx}$.

15. The width of a field is $\frac{2}{3}$ of its length. If the width were increased by 5 feet and the length by 10 feet, the area would be increased by 400 square feet. Find the dimensions.

16. Eleven-sixteenths of a certain principle was at interest at 5%, and the balance at 4%. The entire income was $1000. Find the principal.

17. Find the value of x: $\frac{a(x-a)}{x-b} + \frac{b(x-b)}{x-a} = a + b$.

18. Find the time required for the interest at 5% on a sum of money to be equal to the principal.

19. Solve for each letter: $s = \frac{1}{2}g(2t - 1)$.

20. Solve for y: $\frac{1}{y+1} - \frac{1}{1-y} = \frac{1}{y^2 - 1}$.

21. Solve for S, $C = \frac{SE}{Sb + R}$.

22. Find a, p, and t, in $\frac{c}{2a} = \frac{c}{a + prt}$.

23. A man buys lemons for $2. If he had received for that money 50 more lemons, they would have cost him 2 cents less apiece. What was the price per lemon?

24. Find the time in which $75 will amount to $90.75 at 6%?

25. Solve for F in, $C = \frac{5}{9}(F - 32)$. Also find F when $C = 30$.

26. At what time between 4 and 5 o'clock are the hands of a clock at right angles to each other?

27. At what depth (h) in fresh water will the pressure (P) on an object having a total area (A) of 4 square feet, be 1000 lbs.? Water weighs $62\frac{1}{2}$ lbs. per cubic foot and $P = wAH$.

28. At what time between 5 and 6 o'clock will the hands of a clock be together?

29. A man put out $4330 in two investments. On one of them he gained 12% and on the other he lost 5%. If his net gain was $251, what was the amount of each investment?

30. Solve for x: $\frac{2x+1}{2x-16} - \frac{2x-1}{2x+12} = \frac{9x+17}{x^2 - 2x - 48}$.

CHAPTER XI

GRAPHS

88. If we wish to locate a particular point on the floor of a room, the quickest and simplest method is to measure its distance from each of two walls at right angles to each other. Fig. I.

In the same way, if 2 lines are drawn at right angles to each other, on a piece of paper, any point may be located by stating its distance from each of the two lines. This is most conveniently done by using paper, ruled with horizontal and vertical lines, called *graph* paper.

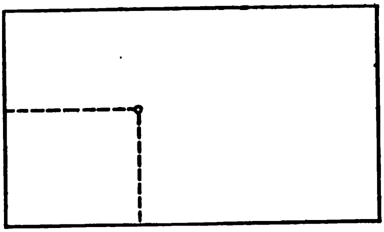


Fig. I.

Thus in Fig. II, any point P, may be located if we know its distance from the vertical line or axis, and its distance from the horizontal line or axis.

Here we can see, that the point P is 7 units from the vertical axis and 8 units from the horizontal axis.

The horizontal distance is called the x of the point and as this distance is equal to the distance OA on the horizontal axis, it can be measured along the horizontal axis, which is therefore known as the X axis. Similarly, the vertical distance, called the y of the point, may be measured along the vertical axis and this is the Y axis.

These two distances, which locate any point in a plane surface, or simply a plane, are called the *coördinates* of the point, and the two axes are called the *coördinate axes* or *axes of reference*. As these measurements are usually made from the point O, the intersection of the two axes, this point is called the *origin*.

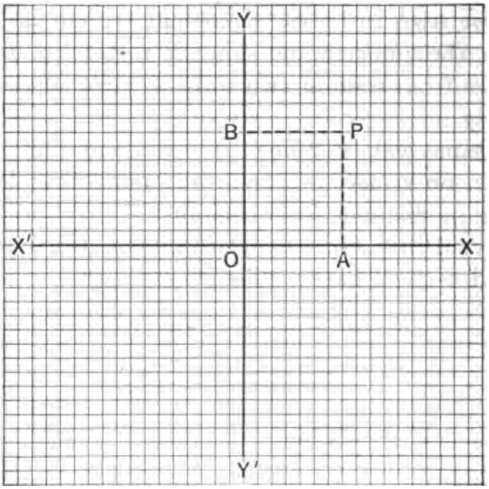


FIG. II.

The horizontal or x distance of a point is also known as the *abscissa* and the vertical or y distance, as the *ordinate* of the point.

If in Fig. III, we imagine a circle drawn from the origin O as a centre, the two axes will divide the circle into 4 equal parts or quadrants. These quadrants are numbered I, II, III and IV, in a direction contrary to the hands of a clock.

In giving the coördinates of a point, the abscissa is always mentioned first. Thus, the point 3, 4, has an abscissa equal to 3 units and an ordinate of 4 units. If we attempted to locate this point, we could find 4 points, each of which would be 3 units from the vertical axis and 4 units from the horizontal axis, one point in each of the 4 quadrants. To overcome this difficulty, all abscissas measured to the right of the Y axis are called + and abscissas measured to the left −, while ordinates measured above the X axis are + and those below the X axis are −. Then, only one point in the first quadrant will have the coördinates 3, 4, as a point in any other quadrant will have one or both of its coördinates negative. Thus, in the second quadrant the abscissa is −, in the 3d quadrant both coördinates are −, and in the 4th quadrant, the ordinate is −.

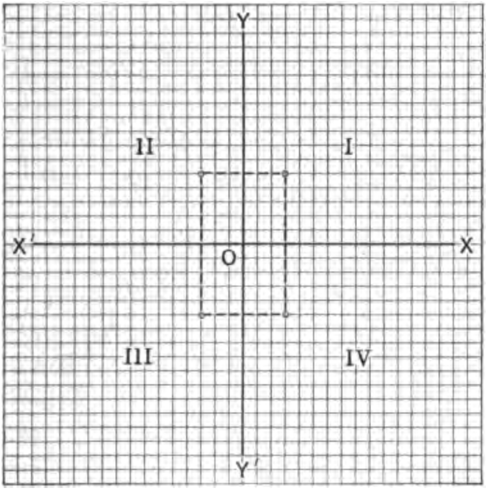


Fig. III.

Locating a point, when its coördinates are known, is called *plotting* the point.

Plot the following points:

1. − 2, 3.
2. 3, − 4.
3. − 2, − 4.
4. 4, 5.
5. 0, − 2.
6. 3, 0.
7. − 4, 0.
8. 0, 3.
9. 0, 0.

It is evident that a point, whose abscissa is 0, must lie on the Y axis, and similarly, a point, whose ordinate is 0, must lie on the X axis.

89. If a point moves across the plane, its path will be a line whose character will depend upon the way in which the point moves. It may be a straight line, a curved line or a broken line. In any case, as the point moves, the values of its coördinates will change, as the distances of the point from the two axes are continually changing. This change in its coördinates may take place according to some prescribed rule or condition. Thus, a point may move so that the sum of its abscissa and ordinate will always equal 3. This condition can be expressed algebraically by the equation $x + y = 3$.

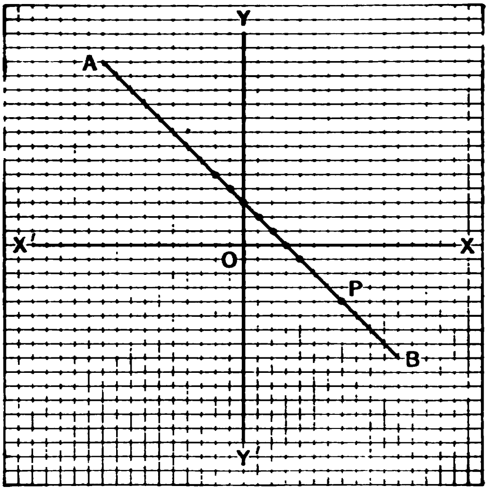


Fig. IV.

Solving this equation of *condition* for y, we have $y = 3 - x$.

If we assign any value to x, this equation gives us the corresponding value of y.

Thus if $x = 1$, $y = 2$.

Plot this point, as shown in Fig. IV.

Similarly, we find the following corresponding values of x and y.

x	y
1	2
2	1
3	0
4	-1
0	3
-1	4
-2	5

Plotting these points, we find that a point, moving so as to obey this condition, $x + y = 3$, will have as its path, a straight line drawn through these points. Any point on this line will have, as coördinates, values of x and y that will satisfy this equation of condition. Thus, the point P, has $x = 7$, $y = -4$, whose sum is 3. The line AB, thus plotted is called the **graph** of the equation $x + y = 3$.

As x and y in this equation may vary in value, they are called **variable quantities** or simply **variables.** The number, 3, does not change in value and is therefore called a **constant quantity** or simply a **constant.**

As the graph of every simple equation, or equation of the first degree, is a straight line, this type of equation is called a **linear equation.**

EXERCISE 59

Plot the graphs of the following equations:

1. $x + y = 4$.
2. $x - y = 2$.
3. $2x + y = 3$.
4. $x - 5y = 5$.
5. $2x + 3y = 12$.
6. $5x + 2y = 10$.
7. $4x = y - 4$.
8. $3x = 6 + 2y$.
9. $3y = 6 + x$.
10. $5x - 3y = 15$.

90. When 2 points of a line are found, these 2 points will fix the position of the line or *determine* the line. Hence to plot the graph of a linear equation, it is only necessary to find two points of the line. The simplest method is to find the two points where the line crosses the two axes.

Thus, plot the graph of $3x - 4y = 24$. Fig. V.

Any point on the X axis will have $y = 0$, hence let $y = 0$ in the equation.

Then $3x = 24$

$x = 8$

Any point on the Y axis will have $x = 0$, hence let $x = 0$.

Then $-4y = 24$

$y = -6$

And AB is the required graph.

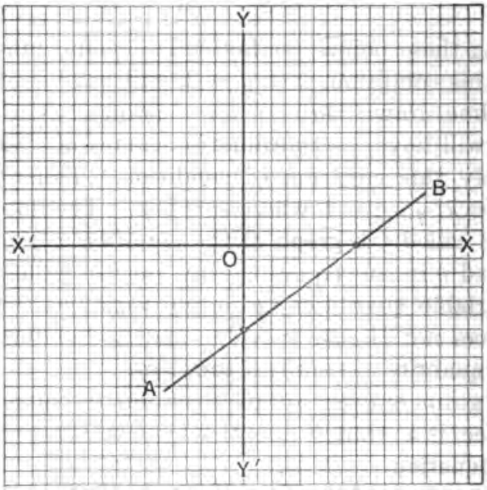


Fig. V.

EXERCISE 60

Plot the graphs of the following equations by locating the points in which the line cuts the two axes.

1. $2x + y = 12$.
2. $7x - 2y = 14$.
3. $5x + y = 10$.
4. $x - y = 3$.
5. $11x - 2y = 22$.
6. $x + 2y = -18$.
7. $5x + 4y = 20$.
8. $2x - 3y = 18$.
9. $3x + 4y = 12$.
10. $4x + y = 20$.

In the following equations, the graph does not cut the axes in two points and therefore any two points must be located by the first method.

11. $2x + y = 0.$
12. $5x - y = 0.$
13. $x + 3y = 0.$
14. $x = 3.$
15. $y = 4.$

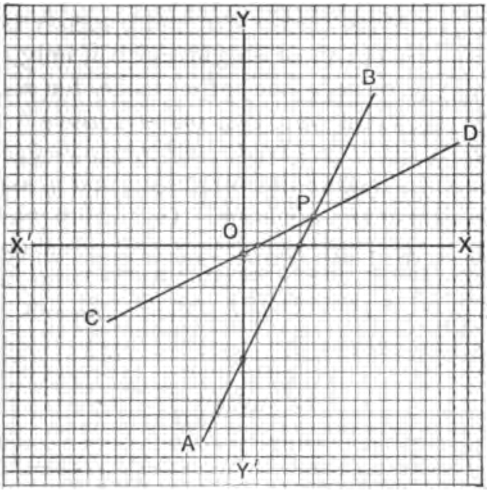


Fig. VI.

91. If, instead of a single equation of condition, two such equations are given and the point must obey both conditions at the same time, the equations are called **simultaneous.**

Thus, the given simultaneous equations, $2x - y = 8$ and $x - 2y = 1$, tell us, first that the point must move, so that twice the x of the point minus the y of the point must always equal 10, and secondly, that, the x of the point minus twice the y of the point must equal 1.

Plotting the graph of the first equation, $2x - y = 8$, we find the line AB in Fig. VI.

Plotting the graph of the second equation, $x - 2y = 1$, we find the line CD.

It is evident that every point on the graph AB, satisfies the condition expressed by the first equation and every point on the graph CD satisfies the condition expressed by the second equation. In order to satisfy both conditions at the same time, the point must lie on both graphs and therefore must be the point at which these two lines intersect.

P, the point of intersection has the coördinates $x = 5$ and $y = 2$ and these values of x and y satisfy both equations. Two straight lines can intersect in only one point, hence there can be only one set of values for x and y that will satisfy two simple simultaneous equations.

A single equation in x and y, as $x + y = 5$, may have any number of values of the variables x and y, and hence is called an *indeterminate* equation. On the other hand, when two simultaneous equations are given, the values of x and y may be found as above. Hence, we have an easy graphical method for solving simple simultaneous equations containing two unknowns.

EXERCISE 61

Solve graphically:

1. $\left.\begin{aligned} x + y &= 5 \\ 2x - y &= 1 \end{aligned}\right\}.$

2. $\left.\begin{aligned} x - y &= 4 \\ x + 2y &= 1 \end{aligned}\right\}.$

3. $\left.\begin{aligned} x + y &= 2 \\ 2x - y &= -8 \end{aligned}\right\}.$

4. $\left.\begin{aligned} 2x - 4y &= 2 \\ 3y + 2x &= 0 \end{aligned}\right\}.$

5. $\left.\begin{aligned} 4x + 7y &= 6 \\ 3x - y &= 17 \end{aligned}\right\}.$

6. $\left.\begin{aligned} x - y &= -1 \\ 3x + y &= 13 \end{aligned}\right\}.$

7. $\left.\begin{aligned} 2x - 3y &= 2 \\ 3x - 2y &= -7 \end{aligned}\right\}.$

8. $\left.\begin{aligned} 2x + y &= 3 \\ x - 4y &= 24 \end{aligned}\right\}.$

9. $\left.\begin{aligned} 5x + 6y &= -4 \\ x - 3y &= -5 \end{aligned}\right\}.$

10. $\left.\begin{aligned} 6x + 7y &= -13 \\ 7x + 6y &= 0 \end{aligned}\right\}.$

When the point of intersection does not fall on an intersection of the ruled lines, the values of x and y must be estimated in fractions and can only be found approximately.

11. $\left.\begin{aligned} 5x - 4y &= 10 \\ 2x + 3y &= 9 \end{aligned}\right\}.$

12. $\left.\begin{aligned} 2x - 7y &= 14 \\ 3x + 5y &= 15 \end{aligned}\right\}.$

13. $\left.\begin{aligned} x + 8y &= 20 \\ 5x - 2y &= 10 \end{aligned}\right\}.$

14. $\left.\begin{aligned} 4x - 9y &= 18 \\ 5y + 2x &= 20 \end{aligned}\right\}.$

15. $\left.\begin{aligned} 2x + y &= -8 \\ 3y + 2x &= 0 \end{aligned}\right\}.$

16. $\left.\begin{aligned} 6x + 7y &= -13 \\ x - 3y &= -5 \end{aligned}\right\}.$

If the graphs of two equations do not intersect, these equations express conditions that cannot be true at the same time and hence the equations are not true simultaneous equations.

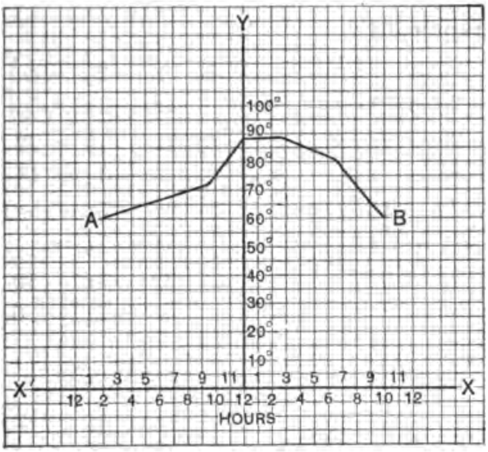


FIG. VII.

92. From what has been said so far, it is clear that a graph shows how two variables change in value. Hence, such a diagram may be used whenever we wish to show the variation of two related quantities. A common instance is a temperature graph, showing the variation in temperature as the time varies. In this case, horizontal units represent units of time and vertical units represent degrees of temperature. See Fig. VII. Let each horizontal unit represent 1 hour and each vertical unit 5 degrees of temperature. The line *AB* shows how the temperature varies from 2'oclock A.M. to 10 P.M.

At any time of day, the vertical distance shows the temperature at that time. Thus at 2 A.M. the temperature was 60°, while at noon, the temperature had risen to about 87°. After reaching a maximum of almost 90° shortly before 3 o'clock P.M. the temperature fell steadily until at 10 A.M. it was again down to 60°.

The student should have no difficulty in finding similar graphs in current magazines.

REVIEW EXERCISE XI

Solve graphically:

1. $\left.\begin{aligned} 2x + y &= -1 \\ x - 2y &= 17 \end{aligned}\right\}.$

2. $\left.\begin{aligned} 5x + 2y &= 11 \\ 3x + 4y &= 1 \end{aligned}\right\}.$

3. $\left.\begin{aligned} 3x - 2y - 16 &= 0 \\ 3y + 4x &= 10 \end{aligned}\right\}.$

4. $\left.\begin{aligned} 5x + 2y &= 9 \\ 2x + 3y &= 8 \end{aligned}\right\}.$

5. $\left.\begin{aligned} 3x + 4y &= 35 \\ 6x - 5y &= 44 \end{aligned}\right\}.$

6. $\left.\begin{aligned} 2x + 7y &= 8 \\ 3x + 9y &= 9 \end{aligned}\right\}.$

7. $\left.\begin{aligned} 2x + 3y &= 24 \\ 5x - 4y &= 14 \end{aligned}\right\}.$

8. $\left.\begin{aligned} x - 2y &= 4 \\ 2x - y &= 5 \end{aligned}\right\}.$

9. $\left.\begin{aligned} 5x + y &= 22 \\ x + 5y &= 14 \end{aligned}\right\}.$

10. $\left.\begin{aligned} 2x - 5y &= -16 \\ 3x + 7y &= 5 \end{aligned}\right\}.$

11. $\left.\begin{aligned} 2x - 3y &= 10 \\ 5x + 2y &= 6 \end{aligned}\right\}.$

12. $\left.\begin{aligned} y &= 4x \\ x - y &= 3 \end{aligned}\right\}.$

13. $\left.\begin{aligned} 2x + 3y &= 12 \\ x - \frac{y}{2} &= 2 \end{aligned}\right\}.$

14. $\left.\begin{aligned} x - y &= -5 \\ x + 3y &= 3 \end{aligned}\right\}.$

15. $\left.\begin{aligned} 3x - 4y &= 26 \\ x - 8y &= 22 \end{aligned}\right\}.$

CHAPTER XII

SIMULTANEOUS SIMPLE EQUATIONS

93. As has been shown in the previous chapter, simple simultaneous equations may be solved graphically by finding a point common to the two graphs.

The algebraic solution also depends upon a combination of the two equations, such, that one of the two unknown numbers may be made to disappear, leaving an equation, containing only one unknown, which may be solved by the usual method.

Elimination is the name given to the process of making one of the unknown numbers disappear.

There are three general methods for eliminating one of the unknowns.

ELIMINATION BY ADDITION OR SUBTRACTION

1. Solve $\begin{cases} 2x - y = 8 & (1) \\ 3x + 2y = 19 & (2) \end{cases}$

Multiplying equation ① by 2,

$$4x - 2y = 16 \quad ③$$

Since the terms containing y in equations ② and ③ have equal coefficients but opposite signs, they will cancel each other when the equations are added.

Therefore adding ② and ③, $7x = 35$

Divide by 7, $x = 5$

Substituting this value of x in equation ②,

$$15 + 2y = 19$$

$$2y = 4$$

and $y = 2$

Hence $x = 5$ and $y = 2$.

2. Solve $\begin{cases} 2x + 3y = 3 & (1) \\ 3x + 2y = 7 & (2) \end{cases}$

To eliminate y, their coefficients must be made equal.

Multiply equation ① by 2, and equation ② by 3,

$$4x + 6y = 6 \quad ③$$

$$9x + 6y = 21 \quad ④$$

Since the y terms have like signs, we eliminate by subtraction. Subtract ③ from ④,

$$\begin{aligned} 5x &= 15 \\ \therefore x &= 3 \end{aligned}$$

Substitute this value of x in ①,

$$\begin{aligned} 6 + 3y &= 3 \\ 3y &= -3 \\ \therefore y &= -1 \end{aligned}$$

Hence $x = 3$ and $y = -1$.

TO ELIMINATE BY ADDITION OR SUBTRACTION

Rule.—*Multiply the equations by numbers that will make the coefficients of one of the unknowns numerically equal.*

Eliminate this unknown by adding the resulting equations if the equal coefficients have unlike signs, or by subtraction if their signs are alike.

It is usual to select that unknown for elimination which will require the smallest multipliers. These multipliers can readily be found if the student will note that the equal coefficients will be the least common multiple of the two original coefficients. Thus in the preceding problem, the L. C. M. of the coefficients of y is 6.

EXERCISE 62

Solve by addition or subtraction:

1. $\left.\begin{aligned} x + y &= 5 \\ 2x - y &= 1 \end{aligned}\right\}.$

2. $\left.\begin{aligned} x - y &= 4 \\ x + 2y &= 1 \end{aligned}\right\}.$

3. $\left.\begin{aligned} x + y &= 2 \\ 2x - y &= -8 \end{aligned}\right\}.$

4. $\left.\begin{aligned} 2x - 4y &= 2 \\ 3y + 2x &= 0 \end{aligned}\right\}.$

5. $\left.\begin{aligned} 4x + 7y &= 6 \\ 3x - y &= 17 \end{aligned}\right\}.$

6. $\left.\begin{aligned} x - y &= -1 \\ 3x + y &= 13 \end{aligned}\right\}.$

7. $\left.\begin{aligned} 2x - 3y &= 2 \\ 3x - 2y &= -7 \end{aligned}\right\}.$

8. $\left.\begin{aligned} 2x + y &= 3 \\ x - 4y &= 24 \end{aligned}\right\}.$

9. $\left.\begin{aligned} 5x + 6y &= -4 \\ x - 3y &= -5 \end{aligned}\right\}.$

10. $\left.\begin{aligned} 6x + 7y &= -13 \\ 7x + 6y &= 0 \end{aligned}\right\}.$

11. $\left.\begin{aligned} 5x - 4y &= 10 \\ 2x + 3y &= 9 \end{aligned}\right\}.$

12. $\left.\begin{aligned} 2x - 7y &= 14 \\ 3x + 5y &= 15 \end{aligned}\right\}.$

ELIMINATION BY SUBSTITUTION

94. Solve: $\left\{\begin{aligned} 7x + 3y &= 16 \quad (1) \\ 5x + 2y &= 13 \quad (2) \end{aligned}\right.$

Finding the value of y in terms of x in equation ①,

$$3y = 16 - 7x$$

$$y = \frac{16 - 7x}{3} \quad ③$$

Substitute this value for y in equation ②,

$$5x + 2\left(\frac{16 - 7x}{3}\right) = 13 \quad ④$$

This equation ④ contains only one unknown, x, and can therefore be solved in the usual way

$$5x + \frac{32 - 14x}{3} = 13$$

Clear of fractions, $15x + 32 - 14x = 39$

Transpose, $15x - 14x = 39 - 32$

$$x = 7$$

Substitute this value of x in equation ③,

$$y = \frac{16 - 49}{3}$$

$$y = \frac{-33}{3} = -11$$

Therefore $x = 7$ and $y = -11$.

TO ELIMINATE BY SUBSTITUTION

Rule.—*Find the value of one of the unknown numbers, in terms of the other, in one of the equations and substitute this value in the other equation.*

Solve the resulting equation.

EXERCISE 63

Solve, by substitution:

1. $\left.\begin{aligned} x - 4y &= 14 \\ y - x &= 4 \end{aligned}\right\}.$

2. $\left.\begin{aligned} x + y &= 10 \\ 7x - 6y &= 34 \end{aligned}\right\}.$

3. $\left.\begin{aligned} 3x - 2y &= 10 \\ x + y &= 70 \end{aligned}\right\}.$

4. $\left.\begin{aligned} 4x - 3y &= 26 \\ 8x - y &= 22 \end{aligned}\right\}.$

5. $\left.\begin{aligned} 5x - 3y &= 7 \\ x - y &= 3 \end{aligned}\right\}.$

6. $\left.\begin{aligned} 3x - y &= 1 \\ 5x - 3y &= -9 \end{aligned}\right\}.$

7. $\left.\begin{aligned} 3x + y &= 17 \\ 2x - 3y &= -7 \end{aligned}\right\}.$

8. $\left.\begin{aligned} x + 4y &= 10 \\ y - x &= 5 \end{aligned}\right\}.$

9. $\left.\begin{aligned} 7x - 3y &= 18 \\ 2x - 5y &= 1 \end{aligned}\right\}.$

10. $\left.\begin{aligned} x - 15y &= -3 \\ 4x - 15y &= 3 \end{aligned}\right\}.$

11. $\left.\begin{aligned} x + 3y &= 1 \\ y - 3x &= 36 \end{aligned}\right\}.$

12. $\left.\begin{aligned} 18x - 7y &= 0 \\ 4x - 3y &= -26 \end{aligned}\right\}.$

ELIMINATION BY COMPARISON

95. Solve $\begin{cases} 4x - 13y = 5 & (1) \\ 3x + 11y = -17 & (2) \end{cases}$

Find the value of x in terms of y,

From equation ①, $4x = 5 + 13y$

$$x = \frac{5 + 13y}{4} \quad ③$$

From equation ②, $3x = -17 - 11y$

$$x = \frac{-17 - 11y}{3} \quad ④$$

As the values of x must be equal, equate these values of x,

$$\frac{5 + 13y}{4} = \frac{-17 - 11y}{3}$$

Clear of fractions, $3(5 + 13y) = 4(-17 - 11y)$

Solve this equation, $15 + 39y = -68 - 44y$

Transpose, $39y + 44y = -68 - 15$

$$83y = -83$$

$$y = -1$$

Substitute this value of y in equation ③,

$$x = \frac{5 - 13}{4} = \frac{-8}{4} = -2$$

TO ELIMINATE BY COMPARISON

Rule.—*Find the value of one unknown number in terms of the other in each equation.*

Equate these two values and solve the resulting equation.

EXERCISE 64

1. $\left.\begin{array}{r} 2x - 3y = 10 \\ x + y = 70 \end{array}\right\}.$

2. $\left.\begin{array}{r} 3x - 2y = 10 \\ 2x + 5y = 6 \end{array}\right\}.$

3. $\left.\begin{array}{r} x + 5y = 22 \\ 5y + x = 14 \end{array}\right\}.$

4. $\left.\begin{array}{r} 3x + 2y = 24 \\ 3x - 5y = 18 \end{array}\right\}.$

5. $\left.\begin{array}{r} 5x + 3y = 14 \\ 3x - 2y = 3 \end{array}\right\}.$

6. $\left.\begin{array}{r} 3x + 2y = 36 \\ 5x - 9y = 23 \end{array}\right\}.$

7. $\left.\begin{array}{r} 2x + 7y = 8 \\ x + 3y = 3 \end{array}\right\}.$

8. $\left.\begin{array}{r} 4x + 6y = 19 \\ 6x - 4y = 9 \end{array}\right\}.$

9. $\left.\begin{array}{r} 4x + 3y = 34 \\ 11x + 5y = 87 \end{array}\right\}.$

10. $\left.\begin{array}{r} 3x - 8y = 44 \\ 5x - 7y = 29 \end{array}\right\}.$

11. $\left.\begin{array}{r} 5x - 6y = 33 \\ x + y = 11 \end{array}\right\}.$

12. $\left.\begin{array}{r} 11x + 3y = 67 \\ 3x - 5y = 5 \end{array}\right\}.$

SIMULTANEOUS FRACTIONAL EQUATIONS

96. Simultaneous equations involving fractions are usually simplified before applying any of the methods of elimination.

Solve
$$\left\{\begin{aligned} \frac{2x-y-3}{4}-\frac{x-2y-3}{3}&=4 \quad (1)\\ 4+\frac{4x-3y-3}{4}-\frac{2x-4y+9}{3}&=0 \quad (2)\end{aligned}\right.$$

Simplify ① by mult. by L. C. M. 12,

$$3(2x-y-3)-4(x-2y-3)=48$$
$$6x-3y-9-4x+8y+12=48$$

Transpose, $6x-4x-3y+8y=48+9-12$

Collect like terms, $2x+5y=45$ ③

Similarly simplify equation ②,

$$48+3(4x-3y-3)-4(2x-4y+9)=0$$
$$48+12x-9y-9-8x+16y-36=0$$
$$12x-8x-9y+16y=36+9-48$$
$$4x+7y=-3 \text{ ④}$$

Eliminate x between ③ and ④ by the method of subtraction.

Multiply ③ by 2,

$$\begin{aligned} 4x+10y&=90 \quad \text{⑤}\\ 4x+7y&=-3 \quad \text{④}\\ \hline 3y&=93\\ y&=31\end{aligned}$$

Subtract,

Substitute this value of y in ③

$$2x+155=45$$
$$2x=45-155=-110$$
$$x=-55$$

$\therefore x=-55$ and $y=31$.

After the equation have been simplified, any one of the three methods of elimination may be applied.

EXERCISE 65

1. $\left.\begin{aligned}\frac{x}{3}+\frac{y}{2}&=7\\ \frac{x}{2}+\frac{y}{3}&=8\end{aligned}\right\}.$

2. $\left.\begin{aligned}\frac{2x}{5}+\frac{3y}{4}&=38\\ \frac{3x}{4}-\frac{2y}{5}&=1\end{aligned}\right\}.$

3. $\left.\begin{aligned}\frac{x+y}{2}+\frac{x-y}{3}&=11\\ \frac{x+y}{3}+\frac{x-y}{2}&=9\end{aligned}\right\}.$

4. $\left.\begin{aligned}\frac{x-4}{2}-\frac{3x-4y}{6}&=2\\ \frac{y+3}{3}+\frac{3x-2y}{4}&=15\end{aligned}\right\}.$

5. $$\left.\begin{aligned} 2y - \frac{2y - 3x}{5} &= 11 \\ \frac{3y - 2x}{5} + 3x &= 16 \end{aligned}\right\}.$$

6. $$\left.\begin{aligned} \frac{5x}{12} - \frac{7x - 4y}{6} &= \frac{7}{2} \\ \frac{5x - 6y}{12} + \frac{5y}{6} &= \frac{13}{2} \end{aligned}\right\}.$$

7. $$\left.\begin{aligned} \frac{1}{4}x + \frac{1}{6}y &= \frac{7}{2} \\ \frac{1}{6}x + \frac{1}{8}y &= \frac{5}{2} \end{aligned}\right\}.$$

8. $$\left.\begin{aligned} \frac{x}{4} - \frac{y}{6} - \frac{1}{2} &= 0 \\ \frac{2x - 1}{4} - \frac{3y - 1}{6} &= \frac{5}{12} \end{aligned}\right\}.$$

9. $$\left.\begin{aligned} \frac{x}{4} + 8y &= 30 \\ \frac{x}{3} + \frac{4y}{3} &= 12 \end{aligned}\right\}.$$

10. $$\left.\begin{aligned} x + \frac{y - 1}{4} &= 3 \\ 4x + \frac{y - 1}{4} &= 9 \end{aligned}\right\}.$$

11. $$\left.\begin{aligned} \frac{3}{x + y} - \frac{1}{x - 1} &= 0 \\ \frac{3}{y - x} - 3 &= 0 \end{aligned}\right\}.$$

12. $$\left.\begin{aligned} \frac{7 + y}{5} - \frac{2y - x}{4} &= 3x - 5 \\ \frac{5x - 7}{2} + \frac{4y - 3}{6} &= 18 - 5y \end{aligned}\right\}.$$

13. $$\left.\begin{aligned} 2x &= y \\ \frac{x + y}{5} + \frac{y - x}{5} &= \frac{2y - x}{8} + \frac{4x - y}{5} + 1 \end{aligned}\right\}.$$

14. $$\left.\begin{aligned} \tfrac{1}{2}(3y - x - 1) - \tfrac{1}{4} &= \tfrac{3}{4}(x - 1) - y \\ \tfrac{1}{5}(7x + 24) - \tfrac{2}{5}(4y + 3x) &= 0 \end{aligned}\right\}.$$

15. $$\left.\begin{aligned} \frac{5x + 3y}{2y - 3} + \frac{6y + 9}{2} - 6\tfrac{1}{2} &= 3y + 4 \\ \frac{6y - 3x}{2(x - 4)} - 4 &= \frac{4x - 9}{5} - \frac{8x + 7}{10} \end{aligned}\right\}.$$

16. $$\left.\begin{aligned} \frac{6x - 10y}{3} &= \frac{31}{6} + \frac{2x - 8y - 9}{6} \\ \left(\frac{2x}{7} + \frac{y}{2} + 2\frac{2}{3}\right) - \frac{5}{3} &= \left(8x - \frac{y}{4} - 50\right) \end{aligned}\right\}.$$

17. $$\left.\begin{aligned}\frac{3x+5}{4}-\frac{9x-4}{6}&=\frac{4y-5x+3}{2y-x}\\ \frac{8y+3}{2}-\frac{12y-2}{3}&=\frac{6x-2y}{7-x}\end{aligned}\right\}.$$

18. $$\left.\begin{aligned}\frac{12x+7y}{5y-3x}&=1\\ \frac{18x-7y}{6y+10}&=2\end{aligned}\right\}.$$

19. $$\left.\begin{aligned}1-\frac{3x}{x-y}&=\frac{x+y}{x-y}\\ \frac{7x-3y}{46}&=\frac{3}{2}\end{aligned}\right\}.$$

20. $$\left.\begin{aligned}\frac{8x+14}{3}-\frac{17+8x}{3}&=\frac{8y-10x}{2x+8}\\ \frac{10x-53}{16}+\frac{4x-13-6y}{4x-6y}&=\frac{5x-12}{8}\end{aligned}\right\}.$$

97. Simultaneous fractional equations, whose denominators are simple expressions containing the unknown numbers, are usually solved by applying elimination *before* clearing the equations of fractions.

1. Solve $$\begin{cases}\dfrac{5}{x}+\dfrac{2}{y}=19 & (1)\\[2ex] \dfrac{3}{x}-\dfrac{8}{y}=-7 & (2)\end{cases}$$

Since the denominators of the corresponding x and y terms are equal, the terms themselves become equal if the numerators are made equal.

Thus, multiplying equation ① by 4,

$$\frac{20}{x}+\frac{8}{y}=76 \; \textcircled{3}$$

Adding ② and ③, the y terms will cancel,

$$\frac{23}{x}=69 \; \textcircled{4}$$

Solving ④ for x,

$$23=69x$$

$$\therefore x=\frac{23}{69}=\frac{1}{3}.$$

Substitute this value of x in equation ①

$$\frac{5}{\frac{1}{3}} + \frac{2}{y} = 19$$

$$15 + \frac{2}{y} = 19$$

$$\frac{2}{y} = 4$$

$$2 = 4y$$

$$\therefore\ y = \frac{2}{4} \text{ or } \frac{1}{2}$$

Hence $x = \frac{1}{3}$ and $y = \frac{1}{2}$

2. Solve $\begin{cases} \dfrac{2}{3x} + \dfrac{1}{2y} = \dfrac{23}{6} \ (1) \\[2ex] \dfrac{4}{5x} + \dfrac{3}{4y} = \dfrac{107}{20} \ (2) \end{cases}$

By multiplying each equation by the L. C. M. of the coefficients in the denominators, the equations are changed to the type of problem 1.

Multiplying ① by 6,

$$\frac{4}{x} + \frac{3}{y} = 23 \ ③$$

Multiplying ② by 20,

$$\frac{16}{x} + \frac{15}{y} = 107 \ ④$$

Multiplying ③ by 5,

$$\frac{20}{x} + \frac{15}{y} = 115 \ ⑤$$

Subtract ④ from ⑤,

$$\frac{4}{x} = 8$$

$$\therefore 4 = 8x$$

$$x = \frac{4}{8} = \frac{1}{2}.$$

Substitute this value of x in ③,

$$\frac{4}{\frac{1}{2}} + \frac{3}{y} = 23$$

$$8 + \frac{3}{y} = 23$$

$$\frac{3}{y} = 15$$

$$3 = 15y$$

$$y = \frac{3}{15} = \frac{1}{5}.$$

Hence $x = \frac{1}{2}$ and $y = \frac{1}{5}$.

EXERCISE 66

Solve:

1. $$\left.\begin{aligned}\frac{1}{x}+\frac{1}{y}&=6\\ \frac{1}{x}-\frac{1}{y}&=3\end{aligned}\right\}.$$

2. $$\left.\begin{aligned}\frac{2}{x}+\frac{3}{y}&=8\\ \frac{5}{x}+\frac{7}{y}&=19\end{aligned}\right\}.$$

3. $$\left.\begin{aligned}\frac{5}{x}+\frac{6}{y}&=16\\ \frac{7}{x}+\frac{8}{y}&=22\end{aligned}\right\}.$$

4. $$\left.\begin{aligned}\frac{1}{2x}+\frac{2}{3y}&=8\\ \frac{3}{4x}+\frac{5}{6y}&=11\end{aligned}\right\}.$$

5. $$\left.\begin{aligned}\frac{1}{3x}+\frac{1}{3y}&=4\\ \frac{1}{3x}-\frac{1}{3y}&=2\end{aligned}\right\}.$$

6. $$\left.\begin{aligned}\frac{5}{x}-\frac{3}{y}&=-2\\ \frac{25}{x}+\frac{1}{y}&=6\end{aligned}\right\}.$$

7. $$\left.\begin{aligned}\frac{3}{2x}-\frac{1}{y}&=-3\\ \frac{5}{2x}+\frac{3}{y}&=23\end{aligned}\right\}.$$

8. $$\left.\begin{aligned}\frac{7}{8x}-\frac{2}{3y}&=-\frac{1}{4}\\ \frac{5}{6x}-\frac{2}{11y}&=\frac{37}{33}\end{aligned}\right\}.$$

9. $$\left.\begin{aligned}\frac{2}{3x}-\frac{3}{4y}&=-\frac{17}{6}\\ \frac{3}{2x}+\frac{4}{3y}&=-\frac{1}{3}\end{aligned}\right\}.$$

In the following problems treat $(x+1)$, $(y+1)$, $(x-1)$, $(y-1)$ just as x and y were treated in the above problems.

10. $$\left.\begin{aligned}\frac{1}{x+1}+\frac{1}{y+1}&=2\\ \frac{1}{x+1}-\frac{1}{y+1}&=1\end{aligned}\right\}.$$

11. $$\left.\begin{aligned}\frac{1}{x-1}+\frac{1}{y+1}&=3\\ \frac{2}{x-1}+\frac{3}{y+1}&=2\end{aligned}\right\}.$$

12. $$\left.\begin{aligned}\frac{5}{x-1}-\frac{4}{y-1}&=10\\ \frac{3}{x-1}-\frac{2}{y-1}&=5\end{aligned}\right\}.$$

13. $$\left.\begin{aligned}\frac{2}{3-x}-\frac{1}{y}&=4\\ \frac{3}{3-x}+\frac{2}{y}&=6\end{aligned}\right\}.$$

14. $$\left.\begin{aligned}\frac{1}{y+2}&=\frac{3}{x}\\ \frac{3}{y+2}&=\frac{5}{x}+8\end{aligned}\right\}.$$

SIMULTANEOUS LITERAL EQUATIONS

98. Literal equations are solved by the same methods as numerical equations. As the known quantities are represented by letters instead of numbers, there will be some differences in details. Sometimes, literal coefficients have some common property or bear certain relations to each other. In this case, it is found convenient to use the same letter with different *accents* or *suffixes.* Thus, ax, $a'x$, or a_1x and a_2x. These are read, *ax, a prime x* or *a sub one x* and *a sub two x*; a, a', a_1 and a_2 are used like different letters.

1. Solve
$$\left.\begin{aligned} ax + by &= c \\ bx + ay &= d \end{aligned}\right\}\begin{matrix}(1)\\(2)\end{matrix}$$

Using the method of addition or subtraction, to eliminate y, the coefficients of y must be made equal.

Multiply ① by a and ② by b,
$$a^2x + aby = ac \quad \textcircled{3}$$
$$b^2x + aby = bd \quad \textcircled{4}$$
Subtracting ④ from ③,
$$a^2x - b^2x = ac - bd$$
Collecting the coefficients of x,
$$(a^2 - b^2)x = ac - bd$$
Divide by the coefficient of x,
$$x = \frac{ac - bd}{a^2 - b^2}$$
To find the value of y, proceed in the same way.

Multiply ① by b and ② by a,
$$abx + b^2y = bc \quad \textcircled{5}$$
$$abx + a^2y = ad \quad \textcircled{6}$$
Subtract ⑤ from ⑥,
$$a^2y - b^2y = ad - bc$$
Collect coefficients of y,
$$(a^2 - b^2)y = ad - bc$$
$$y = \frac{ad - bc}{a^2 - b^2}$$
$$\therefore x = \frac{ac - bd}{a^2 - b^2} \text{ and } y = \frac{ad - bc}{a^2 - b^2}.$$

2. Solve
$$\left\{\begin{aligned} ax - by &= c \ (1) \\ a'x + b'y &= c' \ (2) \end{aligned}\right.$$

Multiply ① by b' and ② by b,

$$ab'x - bb'y = b'c \quad \text{③}$$
$$a'bx + bb'y = bc' \quad \text{④}$$

Add ③ and ④,

$$ab'x + a'bx = b'c + bc'$$

Collect coefficients of x,

$$(ab' + a'b)x = b'c + bc'$$

Divide by coefficient of x,

$$x = \frac{b'c + bc'}{ab' + a'b}.$$

Multiply ① by a' and ② by a,

$$aa'x - a'by = a'c \quad \text{⑤}$$
$$aa'x + ab'y = ac' \quad \text{⑥}$$

Subtract ⑤ from ⑥,

$$ab'y + a'by = ac' - a'c$$

Collect coefficients of y,

$$(ab' + a'b)y = ac' - a'c$$

Divide by coefficient of y,

$$y = \frac{ac' - a'c}{ab' + a'b}$$

$$\therefore x = \frac{b'c + bc'}{ab' + a'b} \text{ and } y = \frac{ac' - a'c}{ab' + a'b}$$

EXERCISE 67

Solve:

1. $\left.\begin{aligned} x + y &= a \\ x - y &= b \end{aligned}\right\}.$

2. $\left.\begin{aligned} mx + ny &= r \\ px + qy &= s \end{aligned}\right\}.$

3. $\left.\begin{aligned} ax + by &= c \\ a'x + b'y &= c' \end{aligned}\right\}.$

4. $\left.\begin{aligned} m(x + y) &= p \\ n(x - y) &= q \end{aligned}\right\}.$

5. $\left.\begin{aligned} x - y &= ab \\ ax &= by \end{aligned}\right\}.$

6. $\left.\begin{aligned} ax - by &= c \\ d(x - y) &= 3 \end{aligned}\right\}.$

7. $\left.\begin{aligned} x - y &= a - b \\ bx + ay &= 2ab \end{aligned}\right\}.$

8. $\left.\begin{aligned} mx + ny &= 1 \\ m_1x + n_1y &= 2 \end{aligned}\right\}.$

9. $\left.\begin{aligned} 2x - 3y &= a + b \\ 3x - 2y &= a - b \end{aligned}\right\}.$

10. $\left.\begin{aligned} \frac{x}{a} + \frac{y}{b} &= 2 \\ ax - by &= 0 \end{aligned}\right\}.$

11. $\left.\begin{aligned} \frac{ax}{a^2 - b^2} + \frac{by}{b^2 - a^2} &= \frac{1}{a - b} \\ bx + ay &= a + b \end{aligned}\right\}.$

12. $$\left.\begin{aligned}\frac{x+a}{y-b}&=\frac{m}{n}\\ ax-by&=c\end{aligned}\right\}.$$

13. $$\left.\begin{aligned}\frac{x}{m}+\frac{y}{n}&=2mn\\ \frac{x}{mn}-\frac{y}{mn}&=m-n\end{aligned}\right\}.$$

14. $$\left.\begin{aligned}\frac{1}{x+a}&=\frac{1}{a-y}\\ \frac{x-y}{x+y}&=a\end{aligned}\right\}.$$

15. $$\left.\begin{aligned}\frac{a}{x}+\frac{b}{y}&=c\\ \frac{b}{x}+\frac{a}{y}&=d\end{aligned}\right\}.$$

16. $$\left.\begin{aligned}\frac{m}{x}-\frac{n}{y}&=\frac{mn}{r}\\ \frac{n}{x}+\frac{m}{y}&=\frac{mr}{n}\end{aligned}\right\}.$$

17. $$\left.\begin{aligned}\frac{2}{ax}-\frac{3}{by}&=4\\ \frac{3}{ax}+\frac{2}{by}&=5\end{aligned}\right\}.$$

18. $$\left.\begin{aligned}\frac{b}{ax}+\frac{a}{by}&=a-b\\ \frac{a}{x}-\frac{b}{y}&=a^2+b^2\end{aligned}\right\}.$$

SIMULTANEOUS EQUATIONS INVOLVING THREE OR MORE UNKNOWN NUMBERS

99. To solve any set of simultaneous equations, there must be at least as many equations as there are unknown numbers. To find the value of any unknown number, we must have an equation containing only one unknown. Hence, if we have three equations, containing three unknown numbers, these must be combined, two at a time, so as to eliminate one of the unknown numbers and thus get two equations, containing only two unknown numbers. These two are then combined, by any of the preceding methods, so as to eliminate another unknown number, producing a single equation containing only one unknown.

The student should always plan his method of solution before proceeding with his work.

1. Solve: $$\left\{\begin{array}{rcrl}x+2y-z&=&-8&(1)\\ 2x-3y+z&=&17&(2)\\ 3x+y-2z&=&-5&(3)\end{array}\right.$$

To eliminate z, add ① and ②,

$$3x-y=9 \quad ④$$

Multiply ② by 2,

$$4x - 6y + 2z = 34 \text{ ⑤}$$

Add ③ and ⑤

$$7x - 5y = 29 \text{ ⑥}$$

To eliminate y, multiply ④ by 5,

$$15x - 5y = 45 \text{ ⑦}$$

Subtract ⑥ from ⑦,

$$8x = 16$$

$$\therefore x = 2$$

Substitute this value of x in ④,

$$6 - y = 7$$

$$- y = 3$$

$$\therefore y = -3$$

Substitute the values of x and y in ②,

$$4 + 9 + z = 17$$

$$\therefore z = 4$$

Hence, $x = 2$, $y = -3$ and $z = 4$.

2. Solve: $\begin{cases} x + y = 5 & (1) \\ y + z = 3 & (2) \\ x + z = 4 & (3) \end{cases}$

Equations of this type, where the coefficients of the unknown numbers are equal, may be solved by a special method, requiring less work.

Add the three equations,

$$2x + 2y + 2z = 12 \text{ ④}$$

Divide by 2,

$$x + y + z = 6 \text{ ⑤}$$

Subtract ② from ⑤,

$$x = 3$$

Subtract ③ from ⑤,

$$y = 2$$

Substract ① from ⑤,

$$z = 1$$

$\therefore x = 3$, $y = 2$, and $z = 1$.

3. Solve $\begin{cases} \dfrac{1}{x} + \dfrac{2}{y} - \dfrac{3}{z} = -4 & (1) \\ \dfrac{2}{x} - \dfrac{1}{z} + \dfrac{2}{v} = -4 & (2) \\ \dfrac{3}{y} + \dfrac{2}{z} - \dfrac{1}{v} = 19 & (3) \\ \dfrac{3}{x} - \dfrac{1}{y} + \dfrac{3}{v} = -3 & (4) \end{cases}$

To eliminate v, multiply ③ by 2,

$$\frac{6}{y}+\frac{4}{z}-\frac{2}{v}=38 \quad ⑤$$

Add ② and ⑤,

$$\frac{2}{x}+\frac{6}{y}+\frac{3}{z}=34 \quad ⑥$$

Multiply ③ by 3,

$$\frac{9}{y}+\frac{6}{z}-\frac{3}{v}=57 \quad ⑦$$

Add ⑦ and ④,

$$\frac{3}{x}+\frac{8}{y}+\frac{6}{z}=54 \quad ⑧$$

To eliminate z, multiply ① by 2,

$$\frac{2}{x}+\frac{4}{y}-\frac{6}{z}=-8 \quad ⑨$$

Add ⑧ and ⑨,

$$\frac{5}{x}+\frac{12}{y}=46 \quad ⑩$$

Add ① and ⑥,

$$\frac{3}{x}+\frac{8}{y}=30 \quad ⑪$$

To eliminate y, multiply ⑩ by 2 and ⑪ by 3,

$$\frac{10}{x}+\frac{24}{y}=92 \quad ⑫$$

$$\frac{9}{x}+\frac{24}{y}=90 \quad ⑬$$

Subtract ⑬ from ⑫,

$$\frac{1}{x}=2$$

Clear of fractions,

$$1=2x$$

$$\therefore x=\tfrac{1}{2}$$

Substitute this value of x in ⑪,

$$\frac{3}{\frac{1}{2}}+\frac{8}{y}=30$$

$$6+\frac{8}{y}=30$$

$$\frac{8}{y}=24$$

$$\frac{1}{y}=3$$

$$1=3y$$

$$\therefore y=\tfrac{1}{3}$$

Substitute the values of x and y in ①,

$$\begin{aligned}
\frac{1}{\frac{1}{2}}+\frac{2}{\frac{1}{3}}-\frac{3}{z} &= -4\\
2+6-\frac{3}{z} &= -4\\
-\frac{3}{z} &= -12\\
\frac{1}{z} &= 4\\
4z &= 1\\
\therefore\quad z &= \tfrac{1}{4}
\end{aligned}$$

Substitute the values of x and y in ④,

$$\begin{aligned}
\frac{3}{\frac{1}{2}}-\frac{1}{\frac{1}{3}}+\frac{3}{v} &= -3\\
6-3+\frac{3}{v} &= -3\\
\frac{3}{v} &= -6\\
\frac{1}{v} &= -2\\
2v &= -1\\
\therefore v &= -\tfrac{1}{2}
\end{aligned}$$

Hence $x = \frac{1}{2}$, $y = \frac{1}{3}$, $z = \frac{1}{4}$ and $v = -\frac{1}{2}$

EXERCISE 68

Solve:

1. $\left\{\begin{aligned} x+y &= 4\\ y+z &= 14\\ x+z &= 7 \end{aligned}\right.$

2. $\left\{\begin{aligned} \frac{1}{x}+\frac{1}{y} &= 8\\ \frac{1}{y}+\frac{1}{z} &= 28\\ \frac{1}{x}+\frac{1}{z} &= 14 \end{aligned}\right.$

3. $\left\{\begin{aligned} x+2y-z &= 1\\ 3x+2y+4z &= 20\\ 2x+5y-3z &= 0 \end{aligned}\right.$

4. $\left\{\begin{aligned} x+y-z &= 6\\ 3x+3y-2z &= 16\\ 5x-10y+3z &= -31 \end{aligned}\right.$

5. $\left\{\begin{aligned} 8x-5y+2z &= 44\\ x-y+z &= 10\\ 13x+9y-3z &= 6 \end{aligned}\right.$

6. $\left\{\begin{aligned} 5x-2y+3z &= 38\\ 3x+9y-8z &= 6\\ 2x-3y-5z &= -56 \end{aligned}\right.$

7. $\left\{\begin{aligned} 3x+3y-4z &= 6\\ 6x-2y+7z &= -46\\ 2x+y-8z &= 40 \end{aligned}\right.$

8. $$\begin{cases} 2x + 5y - 20z = -21 \\ 6x - 3y - 7z = 6 \\ 4x + 8y - 5z = 53 \end{cases}$$

9. $$\begin{cases} 2x - 3y + 4z - 5v = -14 \\ 3x + 2y - 5z + 4v = 12 \\ x + y + 2z - v = 8 \\ 4x - 4y + 3z + 2v = 18 \end{cases}$$

10. $$\begin{cases} u + v + x - y = 2 \\ u + v - x + y = 4 \\ u - v + x + y = 6 \\ v - u + x + y = 8 \end{cases}$$

11. $$\begin{cases} 7x - 11 = 3y \\ 3z + 9 = 7v \\ 7z - 24 = 4y \\ 11x - 13 = 3v \end{cases}$$

12. $$\begin{cases} x + \frac{1}{2}y + \frac{1}{3}z = 20 \\ \frac{1}{2}x + \frac{1}{3}y + \frac{1}{4}z = 13 \\ \frac{1}{3}x + \frac{1}{4}y + \frac{1}{2}z = 17 \end{cases}$$

13. $$\begin{cases} \frac{1}{x} + \frac{2}{y} + \frac{3}{z} = 20 \\ \frac{2}{x} + \frac{3}{y} - \frac{1}{z} = 9 \\ \frac{3}{x} - \frac{1}{y} + \frac{2}{z} = 11 \end{cases}$$

14. $$\begin{cases} \frac{5}{x} - \frac{2}{y} = 14 \\ \frac{2}{x} + \frac{3}{z} = 14 \\ \frac{3}{y} - \frac{2}{z} = 5 \end{cases}$$

15. $$\begin{cases} \frac{3}{x} - \frac{4}{y} + \frac{6}{z} = 9.5 \\ \frac{6}{x} - \frac{8}{y} - \frac{3}{z} = 9 \\ \frac{5}{x} + \frac{12}{y} - \frac{45}{z} = -19.5 \end{cases}$$

PROBLEMS INVOLVING SIMULTANEOUS EQUATIONS

100. In solving certain problems, it is often necessary to use two or more letters to represent the unknown numbers. In all such cases, two or more conditions are stated in the problem, each of which will furnish an equation. The number of equations must equal the number of unknown quantities. If more conditions are given than necessary, some of them may be inconsistent or superfluous. If the number of conditions is less than the number of unknowns, problem cannnot be solved and is called indeterminate.

EXERCISE 69

1. Find two numbers, whose sum is 70, such that twice the greater exceeds three times the smaller by 10.

Let x = the greater number
and y = the smaller number
From the first condition, $x + y = 70$ ①
From the second condition, $2x - 3y = 10$ ②
Solving these two equations, we find that $x = 44$ and $y = 26$.

2. Find two numbers such that twice the greater number equals 7 times the smaller and three times the greater exceeds 5 times the smaller by 11.

3. If the greater of 2 numbers is added to 49 times the smaller, the sum is 51, but if 49 times the greater is added to the smaller, the sum is 99.

4. If 4 times the greater of 2 numbers is divided by the smaller, the quotient is 5 and the remainder 4, but if 5 times the smaller is divided by the greater, the quotient is 3 and the remainder 7. Find the numbers.

5. If A gave B $10, A would have half as much money as B; but if B gave $10 to A, B would have only one-third as much as A. How much has each?

Let x = the number of dollars A has.
and y = the number of dollars B has.
If A gave B $10,
$x - 10$ = the number of dollars A then has
and $y + 10$ = the number of dollars B then has
But as A then has half as much as B,

$$x - 10 = \tfrac{1}{2}(y + 10) \quad ①$$

If B gave A $10,
$x + 10$ = the number of dollars A has
and $y - 10$ = the number of dollars B has
But as B then would have only one third as much as *A*,

$$y - 10 = \tfrac{1}{3}(x + 10) \quad ②$$

Simplifying and solving equations ① and ②, we find $x = 26$ and $y = 22$.

Therefore, A has $26 and B has $22.

6. If A gave B \$10, B would have twice as much money as A; but if B gave A \$5, B would have only $\frac{1}{5}$ as much as A. How much has each?

7. If A gave B \$5, they would have equal sums, but if B gave A \$5, A would have 3 times as much as B. How much has each?

8. If the numerator of a fraction is diminished by 1 and the denominator increased by 1, the value of the fraction becomes $\frac{1}{4}$, but if the numerator is doubled and the denominator diminished by 2, the value of the fraction is $\frac{2}{3}$. Find the fraction.

Let x = the numerator.
and y = the denominator.

Then from the first condition,

$$\frac{x-1}{y+1} = \frac{1}{4} \quad ①$$

and from the second condition,

$$\frac{2x}{y-2} = \frac{2}{3} \quad ②$$

Solving these equations, $x = 7$ and $y = 23$, therefore the fraction is $\frac{7}{23}$.

9. If the denominator of a fraction is increased by 1, the fraction becomes equal to $\frac{1}{2}$, and if both numerator and denominator are diminished by 1, the value of the fraction becomes again $\frac{1}{2}$. Find the fraction.

10. A certain fraction becomes equal to 1, if 1 is added to twice its numerator, and equal to $\frac{1}{6}$ if 2 is added to twice its denominator. Find the fraction.

11. The sum of two fractions, whose numerators are 5 and 7 respectively, is $2\frac{4}{35}$, but if their denominators are interchanged, their sum is 2. Find the fractions.

12. A number is expressed by three digits. The sum of the three digits is 12. The sum of the first and third digits is three times the second digit. If the units' and hundreds' digits are interchanged, the number is diminished by 99. Find the number.

Let x = the hundreds' digit
y = the tens' digit
z = the units' digit

Then, from the first condition,

$$x + y + z = 12 \qquad \text{①}$$

From the second condition,

$$x + z = 3y \qquad \text{②}$$

And as the number equals $100x + 10y + z$, the third condition gives the equation

$$100z + 10y + x = 100x + 10y + z - 99 \qquad \text{③}$$

Solving these three equations, we find that, $x = 5$, $y = 3$, $z = 4$. Therefore the number is 534.

13. A certain number is expressed by 3 digits. The sum of the digits is 10. The tens' digit is equal to the sum of the units' and hundreds' digits. If the tens' and units' digits are interchanged, the number is diminished by 27. Find the number.

14. If a number is divided by the sum of its two digits, the quotient is 4, but if 27 is added to the number, the digits will be reversed. Find the number.

15. A number is expressed by 3 digits. The difference between the units' and tens' digits is equal to the hundreds' digit. If the number is divided by the sum of its digits, the quotient is 37 and the remainder 9. If the digits are reversed the number is increased by 198. Find the number.

16. A man rows 21 miles down a river and back again in 10 hours. He can row 7 miles down the river in the same time that he can row 3 miles up the river. Find the boatman's rate of rowing in still water and the rate of the river's current.

Let x = the number of miles per hour in boatman's rate,
y = the number of miles per hour in river's current.

In going down stream, the rate of the current is added to the rate of rowing and in going up stream, the rate of the current opposes the rate of rowing and is therefore subtracted.

From the first condition,

$$\frac{21}{x + y} + \frac{21}{x - y} = 10 \qquad \text{①}$$

From the second condition,

$$\frac{7}{x+y} = \frac{3}{x-y} \quad ②$$

Solving these two equations, we find that $x = 5$ and $y = 2$. Hence the boatman's rate of rowing in still water is 5 miles per hour and the rate of the current is 2 miles per hour.

17. A boatman rows 14 miles down a river and back again in 9 hours. He can row 7 miles down the river in the same time that he can row 2 miles up the river. Find his rate of rowing in still water and the rate of the current.

18. A boatman can row at the rate of 3½ miles per hour in still water. He finds that he can row down a river 2½ times as fast as up the river. He rows a certain distance down the river and back again in 7 hours. Find the distance and the rate of the current.

19. An air pilot finds that he can fly 5 miles, with the wind, in the same time that he can fly 4 miles against the wind. He can fly 400 miles and back again in 9 hours. Find the time for each trip.

20. A and B can do a piece of work in 3 days, A and C in 4 days and B and C in 6 days. How long will it take each alone to do the work?

Let x = the number of days in which A can do the work.
y = the number of days in which B can do the work.
z = the number of days in which C can do the work.

Then $\frac{1}{x}$, $\frac{1}{y}$ and $\frac{1}{z}$ will equal the parts of the work A, B and C can do in 1 day.

And $\frac{1}{x} + \frac{1}{y}$ = the part A and B can do in 1 day.

But as A and B can do the work in 3 days, they can do ⅓ in 1 day.

$$\text{Hence } \frac{1}{x} + \frac{1}{y} = \frac{1}{3} \quad ①$$

$$\text{and } \frac{1}{x} + \frac{1}{z} = \frac{1}{4} \quad ②$$

$$\text{and } \frac{1}{y} + \frac{1}{z} = \frac{1}{6} \quad ③$$

Solving these three equations, we find that

$$x = \tfrac{24}{5},\ y = 8 \text{ and } z = 24$$

Therefore A can do the work in 4⅘ days, B in 8 days and C in 24 days.

21. A and B can do a piece of work in 1½ days, A and C in 1⅓ days and B and C in 1¼ days. How long will it take each alone to do the work?

22. A cistern has three pipes, A, B, and C. A and B will fill the cistern in 2 hours, A and C in 3 hours, and B and C in 4 hours. How long will it take each pipe alone to fill it?

23. Three men, A, B, and C, can dig a trench in 2 days. A and B could dig it in 3 days, and B and C could dig it in 3½ days. How long would it take each alone to dig the trench?

24. A sum of money at simple interest amounts to $3450 in 3 years, and to $3750 in 5 years. Find the sum and the rate of interest.

Let x = the number of dollars at interest
and y = the number of per cent in the rate.

The interest for one year is found by multiplying the principal by the rate expressed in hundredths, hence the interest for one year is $\frac{xy}{100}$.

From the first condition,

$$x + \frac{3xy}{100} = 3450 \quad ①$$

From the second condition,

$$x + \frac{5xy}{100} = 3750 \quad ②$$

Eliminating the xy terms by the method of subtraction, we find, that $x = 3000$. Substituting this value of x in equation ①, $y = 5$.

Hence the principal is $3000 and the rate 5%.

25. A sum of money at simple interest amounts to $2240 in two years and to $2600 in 5 years. Find the principal and the rate of interest.

26. A sum of money at simple interest amounted to $5125 in 5 months and to $5250 in 10 months. Find the principal and the rate of interest.

27. A man has $9000 to invest. How much must he

invest in 4% bonds and how much in 5% bonds, so that each investment will yield the same income?

28. A man bought 10 horses and 15 mules for $6750. He sold the horses at a profit of 20% and the mules at a profit of 10% and gained $975. Find the average cost of a horse and of a mule.

29. A number of persons bought a boat. If there had been 5 persons more, each would have paid $5 less, but if there had been 10 persons less, each would have paid $25 more. Find the cost of the boat and the number of persons.

30. The distance from New York to Philadelphia is 90 miles. A sets out from Philadelphia at 8 A.M. and B from New York at 8.30 A.M. to meet each other. A motors at the rate of 20 miles per hour and B at the rate of 30 miles per hour but is delayed for 1 hour by an accident. At what time of day and how far from New York will they meet?

31. If the floor of a certain hall were 10 feet narrower and 10 feet longer, the floor space would be 1600 square feet less. But if it were 10 feet wider and 10 feet shorter, the floor space would be 1400 sq. ft. more. Find the dimensions of the floor.

32. If a certain column of troops were arranged with 5 men less in front and ten men more in depth, the number of men would be 250 less; but if arranged with 5 men more in front and 5 men less in depth, the number of men would be 275 more. Find the number of men.

33. A man makes two investments one at 5%, the other at $5\frac{1}{2}$% and his income from both investments is $740. If his first investment were at $5\frac{1}{2}$% and his second at 5%, his income would be reduced $10. Find the sums invested.

34. A huckster bought two kinds of fish, one at 10¢ per lb. the other at 12¢ per lb., the lot costing him $5.60. He sold them all at 18¢ per lb., making a profit of $3.40. How many pounds of each did he buy?

35. A box contains a mixture of 3 lbs. of Mocha coffee and 7 lbs. of Java coffee, and another box contains a mixture of 5 lbs. of Mocha and 20 lbs. of Java. How many lbs. must be taken from each box to make a 4 lb. mixture which will be $\frac{1}{4}$ Mocha?

36. A manufacturer of photographic films has 17 dozen films which he puts up in rolls of 6 exposures and 12 exposures each. If he makes a total of 29 rolls of film, how many of each kind did he make?

37. To see a certain ball game, 500 persons bought admission tickets and 100 persons bought grandstand tickets, the receipts totalling $325. If 100 more of the spectators had bought grandstand tickets, the receipts would have been $25 more. Find the price of each kind of ticket.

38. A wall can be built by 2 men in 6 days. The first man worked alone for 8 days and then the second man alone finished the wall in 3 days. How long would it have taken each man alone to build the wall?

39. A train travels a certain distance in 5 hours. If the speed of the train were 5 miles per hour less, it would take 1 hour more to make the trip. Find the distance and the speed of the train.

40. A coal dealer purchased a lot of coal. If he had bought 10 tons more for the same money, the price per ton would have been 90¢ less; but if he had bought 10 tons less for the same money, the price would have been $1.10 more per ton. Find the number of tons and the price per ton.

REVIEW EXERCISE XII

Solve:

1. $\begin{cases} x + y = 5 \\ y + z = 3 \\ x + z = 6 \end{cases}$

2. $\begin{cases} 2x + 5y = -27 \\ 11x + 6y = -41 \end{cases}$

3. $\begin{cases} 9x - 5y = 52 \\ 8y - 3x = 8 \end{cases}$

4. $\begin{cases} 4s - 13t = 5 \\ 3s + 11t = -17 \end{cases}$

5. $\begin{cases} 3m - 2n = 7 \\ 4m - 7n = -47 \end{cases}$

6. $\begin{cases} 15x + 8y = 1 \\ 10x - 7y = -24 \end{cases}$

7. $\begin{cases} \dfrac{3}{x} - \dfrac{5}{2y} = 16 \\ \dfrac{1}{2x} + \dfrac{4}{y} = -15 \end{cases}$

8. $\begin{cases} 3x - 5y - 2z = 14 \\ 5x - 8y - z = 12 \\ x - 3y - 3z = 1 \end{cases}$

9. $\begin{cases} \dfrac{3}{x} + \dfrac{2}{y} = -\dfrac{31}{40} \\ \dfrac{5}{x} - \dfrac{10}{y} = \dfrac{11}{8} \end{cases}$

10. $\begin{cases} 2u + 2v + 3w = -3 \\ u + v - 5w = 18 \\ 5u - 3v + w = -4 \end{cases}$

11. $\begin{cases} \dfrac{x + y - 2}{x - y} = -\dfrac{1}{3} \\ \dfrac{3x + y - 3}{2y - x} = -\dfrac{1}{11} \end{cases}$

12. $\begin{cases} x + 2y - 3z = 5 \\ 3x - 22y + 6z = 4 \\ 7x - 6y - 3z = 15 \end{cases}$

13. $\begin{cases} \dfrac{3x + 6}{7} - \dfrac{x + 5}{3y - 5} = \dfrac{6x - 2}{14} \\ \dfrac{2y - 3}{6} + \dfrac{3y + 4}{5x - 7} = \dfrac{3y + 5}{9} \end{cases}$

14. $\begin{cases} \dfrac{y - x}{2} - 1 = \dfrac{x - 4}{3} \\ \dfrac{2y + 1}{3} - \dfrac{x + 3}{4} = 0 \end{cases}$

15. $\begin{cases} 5x + 4y + 3z = 35 \\ 4x + 3y + 2z = 25 \\ 3x + 2y - z = 15 \end{cases}$

16. $\begin{cases} \dfrac{x + y}{3} + 3z = 29 \\ \dfrac{2x - y}{2} + 2z = 22 \\ 3x - y = 3(z - 1) \end{cases}$

17. $\begin{cases} x - 2y + 3z = 2 \\ 2x - 3y + z = 1 \\ 3x - y + 2z = 9 \end{cases}$

18. $\begin{cases} bx + ay = abc \\ -x = -dy \end{cases}$

19. $\begin{cases} ax + by = c \\ bx + ay = c \end{cases}$

20. $\begin{cases} \dfrac{1}{x} + \dfrac{1}{y} + \dfrac{1}{z} = 36 \\ \dfrac{1}{x} + \dfrac{3}{y} - \dfrac{1}{z} = 28 \\ \dfrac{1}{x} + \dfrac{1}{3y} + \dfrac{1}{2z} = 20 \end{cases}$

21. $\begin{cases} \dfrac{1}{x} + \dfrac{1}{y} = m \\ \dfrac{1}{x} - \dfrac{1}{y} = n \end{cases}$

22. $\begin{cases} x - y = mn \\ cx - aby = ms \end{cases}$

23. $\begin{cases} \dfrac{a}{x}+\dfrac{b}{y}+\dfrac{c}{z}=a \\ \dfrac{b}{y}-\dfrac{c}{z}+\dfrac{a}{x}=b \\ \dfrac{a}{x}-\dfrac{b}{y}-\dfrac{c}{z}=c \end{cases}$

24. $\begin{cases} \dfrac{1}{x}+\dfrac{1}{y}=\dfrac{1}{a} \\ \dfrac{1}{y}+\dfrac{1}{z}=\dfrac{1}{b} \\ \dfrac{1}{x}+\dfrac{1}{z}=\dfrac{1}{c} \end{cases}$

25. A warehouse will hold 24 boxes and 20 bales. 6 boxes and 14 bales will fill half of it. How many of each alone will it hold?

26. If the greater of two numbers is divided by the smaller, the quotient is 7 and the remainder 4; but if 3 times the greater is divided by twice the smaller, the quotient is 11 and the remainder 4. Find the numbers.

27. If the sides of a rectangular field were each increased by 2 yards, the area would be increased by 220 square yards. If the length were increased and the breadth diminished, each by 5 yards, the area would be diminished by 185 square yards. Find the dimensions of the field.

28. The report of an explosion traveled 1039 feet per second against the wind and 1061 feet per second with the wind. Find the velocity of sound in still air and the velocity of the wind.

29. If 5 is added to both numerator and denominator of a fraction its value is $\frac{1}{2}$. If 3 is subtracted from both, its value is $\frac{1}{4}$. Find the fraction.

30. If two trains 100 miles apart, approach each other, they will meet in 2 hours; but if they run in the same direction, the slow one leading, they will be together in 10 hours. Find the rate of each train.

31. Find 3 numbers such that if each be added to $\frac{1}{3}$ the sum of the others, the results will be 32, 28, and 30, respectively.

32. It is known that the sum of the three angles of a triangle is 180°. The difference between the first and second angles is 10°, and between the second and third angles is 25°. Find the number of degrees in each angle.

33. A certain number of two digits equals six times the sum of its digits, and if the digits were interchanged, the resulting number would be 9 less than the given number. Find the number.

34. A tank has two pumps. If the first is worked 2 hours and the second 3 hours, 1020 cubic feet of water are discharged. But if the first is worked 1 hour and the second $2\frac{1}{2}$ hours, 690 cubic feet are discharged. How many cubic feet can each pump discharge in 1 hour?

CHAPTER XIII

THEORY OF EXPONENTS

101. The rules for the exponents of like letters have been explained for multiplication and division. Thus, to multiply a^2 by a^3, the exponent of a in the product will be the sum of the exponents 2 and 3 of the factors, that is the product will be a^5. Similarly, in dividing a^5 by a^3, the exponent 3 of the divisor is subtracted from 5, the exponent of the dividend, and the quotient will be a^2.

The operation of raising any quantity to a required power is called **Involution.**

Thus, to find the cube of a^2, we must find the product of three factors, each equal to a^2. As each factor contains two factors a, the product will contain six factors a; hence the cube of a^2 equals a^6, that is, the cube of a quantity contains three times as many equal factors as the quantity itself. Similarly, the fifth power would contain five times as many factors. In general, therefore, *any required power of a quantity can be found by multiplying the exponent of that quantity by the exponent of the power to which it is to be raised.*

This rule will hold whether the exponents are numbers or letters.

$$(x^2)^5 = x^{10} \text{ and } (a^m)^n = a^{mn}$$

If the quantity consists of two or more unequal factors, the rule applies to each factor.

Thus, $(a^2b^3)^2 = a^{2\times 2}b^{3\times 2} = a^4b^6$
and $(x^ay^b)^m = x^{a\times m}y^{b\times m} = x^{am}y^{bm}$

The operation of finding any required root of a quantity is called **Evolution.**

Thus, the cube root of a^6 is one of the three equal factors of a^6. Since a^6 contains 6 factors each equal to a, the cube root of a^6 will contain $\frac{1}{3}$ of the total number of six factors, that is, two a factors. Hence the cube root of a^6 equals a^2. To indicate a root, the root sign or *radical sign*, $\sqrt{\ }$, is used, with the proper index placed within the sign. Thus the cube root of a^6 is written $\sqrt[3]{a^6}$.

The rule of exponents for evolution is, therefore,

Any required root of a quantity can be found by dividng the exponent of the quantity by the index of the root.

102. It is evident that of the six operations, *addition, subtraction, multiplication, division, involution* and *evolution*, the only two that can not be performed by means of the exponents, are addition and subtraction.

Multiplication is performed by **addition** of exponents.

Division is performed by **subtraction** of exponents.

Involution is performed by **multiplication** of exponents.

Evolution is performed by **division** of exponents.

The rules for exponents may be briefly stated:

The last four of the six operations are performed by the first four of the exponents.

103. The exponents considered, so far, have been positive integers. In applying the rules for exponents, other forms of exponents frequently arise.

Meaning of a zero exponent.

Whenever a number is subtracted from its equal, the result is zero.

A zero exponent, therefore, is produced whenever an exponent is subtracted from an equal exponent. But subtraction of exponents occurs in division. Hence, whenever a quantity is divided by its equal, the exponent of the quotient will be zero.

Thus, $\frac{a^3}{a^3} = a^{3-3} = a^0 = 1$, since a quantity divided by its equal will yield a quotient equal to 1.

A zero exponent therefore, indicates that a quantity has been divided by its equal. Any quantity with a zero exponent is equal to 1.

Meaning of a negative exponent.

104. If a larger number is subtracted from a smaller number, the result is a negative number.

Hence, whenever a quantity is divided by another quantity having a larger exponent, the quotient will have a negative exponent.

Thus, $\frac{a^2}{a^5} = a^{2-5} = a^{-3}$.

Considering the indicated division $\frac{a^2}{a^5}$ as a fraction, if both numerator and denominator are divided by a^2, the result will be $\frac{1}{a^3}$, and, as such a division does not alter the value of the fraction, $\frac{a^2}{a^5} = a^{-3} = \frac{1}{a^3}$.

Similarly, if both terms of the fraction $\frac{a^5}{a^2}$ are divided by a^5,

$$\frac{a^5}{a^2} = \frac{1}{a^{-3}}$$

or dividing both numerator and denominator by a^2,

$$\frac{a^5}{a^2} = \frac{a^3}{1} = a^3$$

Hence, $a^{-3} = \frac{1}{a^3}$ and $\frac{1}{a^{-3}} = a^3$

A negative exponent, therefore, shows that a quantity has been divided by another quantity having a larger exponent. Also, any **factor,** *having a negative exponent, may be transferred from the numerator to the denominator, or from the denominator to the numerator, provided the sign of its exponent is changed.*

Meaning of a fractional exponent.

105. If a number is divided by another which is not an exact divisor of the first, the quotient will be fractional. Hence, in evolution, if an exponent is divided by the index of the root, and the index is not an exact divisor of the exponent, the resulting exponent will be fractional.

Thus, $\sqrt[3]{a^5} = a^{\frac{5}{3}}$.

The exponent $\frac{5}{3}$ indicates the cube root of the fifth power of a.

Hence, *a fractional exponent indicates the root of a quantity that is not a perfect power of the same degree as the index.*

The numerator of the fractional exponent indicates the power and the denominator the root.

All rules of exponents apply equally to literal and numerical exponents.

EXERCISE 70

Express with fractional exponents:

1. $\sqrt[3]{a^2y^5} + \sqrt[5]{a^3x^4}$.

To extract the cube root, divide the exponents by 3, and to extract the 5th root, divide the exponents by 5.

$$\therefore \sqrt[3]{a^2y^5} + \sqrt[5]{a^3x^4} = a^{\frac{2}{3}}y^{\frac{5}{3}} + a^{\frac{3}{5}}x^{\frac{4}{5}}$$

2. $\sqrt[3]{x^2}$.
3. $\sqrt{a^3}$.
4. $\sqrt[4]{y^3}$.
5. $\sqrt[3]{2ab}$.
6. $\sqrt[4]{ab^2c}$.
7. $\sqrt[5]{16x^2y^3}$.
8. $\sqrt[3]{-27}$.
9. $\sqrt{x} - \sqrt[3]{a^2b^2}$.
10. $\sqrt[4]{8ab} + \sqrt[3]{x^2y^4} - \sqrt[5]{16abc}$.

Express with radical signs:

11. $3a^{\frac{1}{3}} + x^{\frac{1}{2}}y^{\frac{2}{3}}$.

As the denominator indicates the root, each factor is placed under the radical sign with its respective index. Note, that the exponent of 3 is not fractional and also that the index 2 is understood.

$$3a^{\frac{1}{3}} + x^{\frac{1}{2}}y^{\frac{2}{3}} = 3\sqrt[3]{a} + \sqrt{x}\sqrt[3]{y^2}$$

12. $x^{\frac{2}{3}}$.
13. $a^{\frac{1}{2}}b^{\frac{1}{3}}$.
14. $2y^{\frac{3}{4}}$.
15. $x^{\frac{2}{3}}y^{-\frac{3}{4}}$
16. $ab^{\frac{2}{3}}c^{\frac{5}{3}}$.
17. $x^{\frac{1}{3}} + x^{\frac{2}{3}}y^{\frac{1}{3}}$.
18. $2a^{\frac{2}{3}} - 4a^{\frac{3}{4}}bc^{\frac{3}{4}}$.
19. $x^{\frac{3}{4}}y^{\frac{3}{4}} + ab^{\frac{2}{3}}c^{-\frac{3}{4}}$.
20. $3x^{-\frac{1}{3}} - 2^{\frac{1}{2}}y^{\frac{2}{3}}z^{-\frac{1}{4}}$.

Express with positive exponents:

21. $\dfrac{2x^{-\frac{1}{3}}a^2}{3^{-2}y^2b^{-1}}$.

The factors having negative exponents are transferred to the other side of the line of the fraction.

$$\frac{2x^{-\frac{1}{3}}a^2}{3^{-2}y^2b^{-1}} = \frac{2\cdot 3^2a^2b}{x^{\frac{1}{3}}y^2} = \frac{18a^2b}{x^{\frac{1}{3}}y^2}$$

22. $\dfrac{a^{\frac{2}{3}}b^{-1}c^{-2}}{a^{-\frac{1}{2}}xy}$.

The factors containing a are combined by adding their exponents and as these exponents are fractional, they must be changed to a common denominator.

$$\frac{a^{\frac{2}{3}}b^{-1}c^{-2}}{a^{-\frac{1}{2}}xy}=\frac{a^{\frac{2}{3}}a^{\frac{1}{2}}}{bc^2xy}=\frac{a^{\frac{4}{6}+\frac{3}{6}}}{bc^2xy}=\frac{a^{\frac{7}{6}}}{bc^2xy}$$

23. $a^{-\frac{2}{3}}$.
25. $x^{-\frac{1}{2}}$.
27. $\dfrac{a^2}{3b^{-\frac{1}{2}}}$.
29. $\dfrac{a^{\frac{1}{2}}x^{-1}y^{-3}}{a^{-\frac{1}{2}}x^3y^{\frac{1}{2}}}$.
24. $\dfrac{a}{b^{-1}}$.
26. $2x^{-\frac{1}{3}}$.
28. $\dfrac{2x^{-1}}{3^{-3}a^{-1}y^2}$.
30. $\dfrac{3x^2y^{\frac{1}{2}}}{4^{-2}x^{-\frac{1}{2}}y^{-\frac{1}{2}}}$.

Express without denominators:

31. $\dfrac{2a^{-3}b^{-\frac{1}{2}}c^2}{5^{-1}a^2b^{\frac{1}{2}}c^5}=2\cdot 5a^{-3}a^{-2}b^{-\frac{1}{2}}b^{-\frac{1}{2}}c^2c^{-5}=10a^{-5}b^{-1}c^{-3}$.

32. $\dfrac{4a^2b^2}{x^2}$.
35. $\dfrac{abc}{a^{-3}bc^{-4}}$.
38. $\dfrac{3abx}{2^{-1}a^{\frac{1}{2}}b^{\frac{1}{3}}x^{\frac{1}{4}}}$.
33. $\dfrac{a^2b^2}{a^{-1}b^3}$.
36. $\dfrac{2xyz}{3^{-1}xy^2z^{-\frac{1}{2}}}$.
39. $\dfrac{2^{-1}a^{-3}b^{-4}c^{-5}}{2^{-2}a^{-2}b^{-4}c^{-6}}$.
34. $\dfrac{2x^2y^{-1}z^2}{xy^{-1}z}$.
37. $\dfrac{a^{-3}b^2c^{-4}}{axy}$.

Simplify:

40. $(\frac{1}{125})^{\frac{1}{3}}\times(8)^{-\frac{5}{3}}$.

Problems involving numbers are most easily solved, by writing the expressions without denominators and expressing the numbers as powers of some prime number.

$$(\tfrac{1}{125})^{\frac{1}{3}}\times(8)^{-\frac{5}{3}}=(125^{-1})^{\frac{1}{3}}\times(8)^{-\frac{5}{3}}$$

$(125^{-1})^{\frac{1}{3}}$ may be considered a case of involution, hence multiply the exponents.

$(125)^{-\frac{1}{3}}\times(8)^{-\frac{5}{3}}=(5^3)^{-\frac{1}{3}}\times(2^3)^{-\frac{5}{3}}=(5)^{3\times-\frac{1}{3}}\times(2)^{3\times-\frac{5}{3}}=5^{-1}\times 2^{-5}=$
$\frac{1}{5}\times\frac{1}{2^5}=\frac{1}{5}\times\frac{1}{32}=\frac{1}{160}$.

41. $16^{\frac{3}{4}}$.
42. $8^{-\frac{2}{3}}$.
43. $(-25)^{\frac{3}{2}}$.
44. $(8)^{\frac{2}{3}}\times(4)^{-\frac{3}{2}}$.
45. $(81)^{-\frac{3}{4}}\times(4)^{\frac{5}{2}}$.
46. $(\frac{1}{25})^{-\frac{1}{2}}\times(16)^{\frac{3}{4}}$.
47. $(\frac{1}{8})^{-\frac{2}{3}}\times(\frac{1}{16})^{\frac{3}{4}}$.
48. $\dfrac{2^0}{(16)^{-\frac{1}{4}}}\times\dfrac{(8)^{\frac{2}{3}}}{3^0}$.
49. $(\frac{1}{64})^{-\frac{1}{3}}\times(\frac{1}{125})^{\frac{1}{3}}$.

If $a = 2, b = 1, c = 4$, find the numerical value of:

50. $a^{\frac{1}{2}}bc^{\frac{1}{2}}$
51. $ac^{-\frac{1}{2}}$.
52. $ab^3c^{\frac{3}{2}}$.
53. $a^{-2}b^{\frac{1}{2}}c^{\frac{1}{2}}$.
54. $5abc^{-\frac{3}{2}}$.
55. $2(abc)^{-\frac{2}{3}}$.
56. $3(a^{-1}b^2c^2)^{\frac{1}{3}}$.
57. $(a^0bc)^{-\frac{1}{2}}$.
58. $(a^{\frac{3}{2}}b^4c^{-\frac{1}{2}})^{\frac{1}{2}}$.

MULTIPLICATION AND DIVISION

106. Multiplications and divisions of polynomials, introducing negative and fractional exponents, are performed in the usual way, remembering, that in multiplication the exponents are added and that in division, the exponent of the divisor is subtracted from the exponent of the dividend.

In the following problems, the student will find it convenient to avoid radical signs, by using fractional exponents and to change fractions to integral expressions as shown in the preceding exercise.

EXERCISE 71

1. Multiply $\sqrt[3]{a^2} + \sqrt[3]{ab} + \sqrt[3]{b^2}$ by $\sqrt[3]{a} - \sqrt[3]{b}$.

Using fractional exponents,

$$\begin{array}{llll}
a^{\frac{2}{3}} & + a^{\frac{1}{3}}b^{\frac{1}{3}} & + b^{\frac{2}{3}} & \\
a^{\frac{1}{3}} & - b^{\frac{1}{3}} & & \\
\hline
a & + a^{\frac{2}{3}}b^{\frac{1}{3}} & + a^{\frac{1}{3}}b^{\frac{2}{3}} & \\
 & - a^{\frac{2}{3}}b^{\frac{1}{3}} & - a^{\frac{1}{3}}b^{\frac{2}{3}} & - b \\
\hline
a & & & - b
\end{array}$$

Multiply:

2. $x^{\frac{1}{2}} + y^{\frac{1}{2}}$ by $x^{\frac{1}{2}} - y^{\frac{1}{2}}$.
3. $\sqrt{x} + \sqrt{y}$ by $\sqrt{x} + \sqrt{y}$.
4. $x^{\frac{2}{3}} - x^{\frac{1}{3}}y^{\frac{1}{3}} + y^{\frac{2}{3}}$ by $x^{\frac{1}{3}} + y^{\frac{1}{3}}$.
5. $a + b$ by $a^{\frac{1}{2}} - b^{\frac{1}{2}}$.
6. $a^{-2} + a^{-1}b^{-1} + b^{-2}$ by $a^{-1} - b^{-1}$.
7. $x^{-\frac{1}{2}} + y^{-\frac{1}{2}}$ by $x^{-\frac{1}{2}} - y^{-\frac{1}{2}}$.
8. $x^{-2} + x^{-1}y^{-1} + y^{-2}$ by $x^{-2} - x^{-1}y^{-1} + y^{-2}$.
9. $1 + a^{-1} + a^{-2}$ by $1 - a^{-1} + a^{-2}$.
10. $x^{\frac{2}{3}}y^{-\frac{1}{2}} + 2x^{\frac{1}{3}}y^{-\frac{1}{2}} + 3y^{-\frac{3}{2}}$ by $2x^{-\frac{1}{3}}y^{\frac{1}{2}} - 3x^{-\frac{2}{3}}y^{\frac{1}{2}} + 4x^{-1}y^{\frac{3}{2}}$.
11. Divide $x - y$ by $\sqrt[4]{x} - \sqrt[4]{y}$.

$$\begin{array}{l|l} x - y & x^{\frac{1}{4}} - y^{\frac{1}{4}} \\ \underline{x - x^{\frac{3}{4}}y^{\frac{1}{4}}} & \overline{x^{\frac{3}{4}} + x^{\frac{1}{2}}y^{\frac{1}{4}} + x^{\frac{1}{4}}y^{\frac{1}{2}} + y^{\frac{3}{4}}} \\ \quad x^{\frac{3}{4}}y^{\frac{1}{4}} - y & \\ \quad \underline{x^{\frac{3}{4}}y^{\frac{1}{4}} - x^{\frac{1}{2}}y^{\frac{1}{2}}} & \\ \qquad x^{\frac{1}{2}}y^{\frac{1}{2}} - y & \\ \qquad \underline{x^{\frac{1}{2}}y^{\frac{1}{2}} - x^{\frac{1}{4}}y^{\frac{3}{4}}} & \\ \qquad\quad x^{\frac{1}{4}}y^{\frac{3}{4}} - y & \\ \qquad\quad \underline{x^{\frac{1}{4}}y^{\frac{3}{4}} - y} & \end{array}$$

12. Divide $a^{-4} + a^{-2}b^{-2} + b^{-4}$ by $a^{-2} + b^{-2} + a^{-1}b^{-1}$.

$$\begin{array}{l|l} a^{-4} + a^{-2}b^{-2} + b^{-4} & a^{-2} + a^{-1}b^{-1} + b^{-2} \\ \underline{a^{-4} + a^{-3}b^{-1} + a^{-2}b^{-2}} & \overline{a^{-2} - a^{-1}b^{-1} + b^{-2}} \\ \quad - a^{-3}b^{-1} + b^{-4} & \\ \quad \underline{- a^{-3}b^{-1} - a^{-2}b^{-2} - a^{-1}b^{-3}} & \\ \qquad\qquad a^{-2}b^{-2} + a^{-1}b^{-3} + b^{-4} & \\ \qquad\qquad \underline{a^{-2}b^{-2} + a^{-1}b^{-3} + b^{-4}} & \end{array}$$

Divide:

13. $a^{\frac{2}{3}} - b^{\frac{2}{3}}$ by $a^{\frac{1}{3}} - b^{\frac{1}{3}}$.
14. $x^{-3} + y^{-3}$ by $x^{-1} + y^{-1}$.
15. $x^{-\frac{4}{3}} - y^{-\frac{4}{3}}$ by $x^{-\frac{1}{3}} - y^{-\frac{1}{3}}$.
16. $a^{\frac{4}{3}} + a^{\frac{2}{3}}b^{\frac{2}{3}} + b^{\frac{4}{3}}$ by $a^{\frac{2}{3}} - a^{\frac{1}{3}}b^{\frac{1}{3}} + b^{\frac{2}{3}}$.
17. $a + b$ by $a^{\frac{1}{3}} + b^{\frac{1}{3}}$.
18. $2a^{\frac{3}{2}}b^{-1} - 20 + 18a^{-\frac{3}{2}}b$ by $a^{\frac{3}{4}}b^{-\frac{1}{2}} + 2a^{\frac{1}{4}} - 3b^{\frac{1}{2}}$.

REVIEW EXERCISE XIII

1. Find the numerical value of $(\frac{9}{16})^{-\frac{3}{2}}$ and $8^{\frac{2}{3}} \times 4^{-\frac{1}{2}}$.
2. Simplify $\left(-\dfrac{3x^2y^3z^4}{2a^2b^5}\right)^5$.
3. Divide $a + b$ by $a^{\frac{1}{3}} - b^{\frac{1}{3}}$.
4. Express with positive exponents, a^{-3}, $3x^{-3}y^{-2}$.
5. Simplify $8^{-\frac{2}{3}} \times 4^{\frac{1}{2}}$ and $(a^{-\frac{3}{2}}b^3)^{\frac{2}{3}}$.
6. Divide $x - 3x^{\frac{2}{3}} + 3x^{\frac{1}{3}} - 1$ by $x^{\frac{1}{3}} - 1$.
7. Simplify: $27^{-\frac{2}{3}} \times 27^{\frac{4}{3}} \times 9^{-1} \times 8x^0$.
8. Simplify: $\{(a^{\frac{1}{2}}b^{\frac{2}{3}})^{\frac{1}{2}} \div (a^{-\frac{1}{3}}b)^{-2}\}^{\frac{1}{6}}$.
9. Divide $x^{\frac{2}{3}} - 4x^{\frac{1}{3}} + 1 + 6x^{-\frac{1}{3}}$ by $x^{\frac{1}{3}} - 2$.
10. Find the value of: $8^{-\frac{2}{3}} + 2^{-2} - 6^0 + 5y^0 - 9^{\frac{1}{2}}$.
11. Simplify $[\sqrt[3]{x^{\frac{1}{2}}y^{-2}} \div \sqrt[5]{x^{-\frac{1}{3}}y^4}]^{\frac{15}{2}}$.
12. Simplify $\sqrt[2m]{\dfrac{8^0 \cdot 16^{-\frac{1}{4}} \cdot 4^{-2}}{1^{-8} \cdot 4^{-\frac{1}{2}} \cdot 2^{-m-4}}}$.

13. Simplify $16^{-\frac{1}{4}} - \dfrac{8^{-\frac{1}{3}}}{2} + \dfrac{\sqrt[4]{2^3}}{\sqrt{2^{-\frac{1}{2}}}} - \dfrac{\sqrt[5]{2}}{4^{-\frac{2}{5}}}$.

14. Divide $x^2 - 3x^{\frac{5}{3}} + 6x^{\frac{4}{3}} - 7x + 6x^{\frac{2}{3}} - 3x^{\frac{1}{3}} + 1$ by $x^{\frac{2}{3}} - x^{\frac{1}{3}} + 1$.

15. Simplify $81^{-\frac{3}{4}} - \dfrac{3}{27^{-\frac{1}{3}}} + \dfrac{\sqrt[4]{3^3}}{\sqrt{3^{-\frac{1}{2}}}} - \dfrac{\sqrt[5]{3}}{9^{-\frac{2}{5}}}$.

16. Simplify $\sqrt[3]{2 \cdot 4^{-5}} - \dfrac{\sqrt[3]{2}}{2^{-\frac{2}{3}}} + \dfrac{1}{8^{-\frac{1}{3}}} - 128^{-\frac{3}{7}}$.

17. Simplify $\dfrac{2^{n+4} - 2 \cdot 2^n}{2^{-2} \cdot 2^{-n-2}}$.

18. Simplify $\left\{\dfrac{x^{\frac{1}{2}}y^{\frac{4}{3}}}{z^{-\frac{1}{4}}} \times \dfrac{z^4}{x^{-3}y^{-\frac{4}{3}}} \div \dfrac{y^{-2}z^{\frac{1}{4}}}{x^{-\frac{1}{2}}}\right\}^{\frac{1}{3}}$.

19. Simplify $\dfrac{2^n \cdot (2^{n-1})^n \cdot 8 \cdot 2^{-n}}{2^{n+1} \cdot 2^{n-1} \cdot \sqrt{4} \cdot 2^{n^2-4n+1}}$.

20. Simplify: $(a + b) \div (a^{\frac{1}{3}} + b^{\frac{1}{3}}) - (a^{\frac{1}{3}} - b^{\frac{1}{3}})^2$.

CHAPTER XIV

INVOLUTION AND EVOLUTION

INVOLUTION

107. The index law for involution has been explained in the preceding chapter.

As Involution is a form of multiplication in which all the factors are equal to each other, it is evident that,

Any required power of a fraction is found by raising the numerator and denominator to the required power.

$$\text{Thus } \left(\frac{a^2b^2}{xy^3}\right)^3 = \frac{(a^2b^2)^3}{(xy^3)^3} = \frac{a^6b^6}{x^3y^9}.$$

The law of signs in multiplication states that like signs produce a plus product. Hence any pair of equal factors must produce a positive product and any product, consisting of an even number of equal factors, must also be positive. Hence, *Any even power of a quantity will be positive.*

In the case of an odd power, consisting of an odd number of equal factors, if the factors are paired, a single factor will be left over. The product of the pairs of factors will be positive, and this product multiplied by the remaining single factor will yield either a positive or a negative product, according to its sign. Hence, *Any odd power of a quantity will have the same sign as the quantity itself.*

Thus, $(xy)^4 = x^4y^4$ and $(-xy)^4 = x^4y^4$
but $(xy)^3 = x^3y^3$ and $(-xy)^3 = -x^3y^3$

INVOLUTION OF BINOMIALS

108. By actual multiplication, we find, that
$(a + b)^2 = a^2 + 2ab + b^2$
$(a - b)^2 = a^2 - 2ab + b^2$
$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

$$(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$$
$$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$
$$(a-b)^4 = a^4 - 4a^3b + 6a^2b^2 - 4ab^3 + b^4$$

From these results, it will be evident that:

1. The number of terms is one greater than the exponent of the binomial.
2. The exponent of the first term is equal to the exponent of the binomial and the exponent of a, the first term of the binomial, decreases by one in each succeeding term while the exponent of b, the second term of the binomial, increases by one in each succeeding term.
3. The coefficient of the first term is 1.
4. The coefficient of the second term is equal to the exponent of the first term.
5. The coefficient of each succeeding term is found by multiplying the coefficient of the preceding term by the exponent of a and dividing that product by one more than the exponent of b.
6. In finding any power of $a - b$, the terms containing the odd powers of b are negative, that is, the terms are alternately plus and minus.

Applying these rules,

$$(x-y)^5 = x^5 - 5x^4y + 10x^3y^2 - 10x^2y^3 + 5xy^4 - y^5$$

Also
$$\begin{aligned}(2a^2 - 3b^3)^4 &= (2a^2)^4 - 4(2a^2)^3(3b^3) + 6(2a^2)^2(3b^3)^2 - \\ &\qquad 4(2a^2)(3b^3)^3 + (3b^3)^4 \\ &= 16a^8 - 4(8a^6)(3b^3) + 6(4a^4)(9b^6) - \\ &\qquad 4(2a^2)(27b^9) + 81b^{12} \\ &= 16a^8 - 96a^6b^3 + 216a^4b^6 - 216a^2b^9 + 81b^{12}\end{aligned}$$

EXERCISE 72

Raise to the required power:

1. $\left(\frac{2a^2}{3b}\right)^3$.
2. $\left(\frac{3x^{-1}}{2y^{-2}}\right)^4$.
3. $\left(-\frac{2x^{-3}}{3a}\right)^4$.
4. $(a+2)^4$.
5. $(x-2)^5$.
6. $(a^2-1)^6$.
7. $(2x-3)^5$.
8. $(3a^2+b)^4$.
9. $(2x-3y)^6$.
10. $(ab^2+2a^2b)^4$.
11. $(x^{-1}+y^{-1})^5$.
12. $(2a^{-2}-bc^{-1})^4$.
13. $(a-b+c-d)^3$.

Expressions containing 3 or 4 terms may be raised to any required power by the same rules, if the terms of the expression are grouped to form a binomial.

$$(a-b+c-d)^3 = [(a-b)+(c-d)]^3 =$$
$$(a-b)^3 + 3(a-b)^2(c-d) + 3(a-b)(c-d)^2 + (c-d)^3 =$$
$$a^3 - 3a^2b + 3ab^2 - b^3 + 3(a^2 - 2ab + b^2)(c-d) +$$
$$3(a-b)(c^2 - 2cd + d^2) + c^3 - 3c^2d + 3cd^2 - d^3 =$$
$$a^3 - 3a^2b + 3ab^2 - b^3 + 3(a^2c - 2abc + b^2c - a^2d + 2abd - b^2d)$$
$$+ 3(ac^2 - 2acd + ad^2 - bc^2 + 2bcd - bd^2) + c^3 - 3c^2d + 3cd^2 - d^3 =$$
$$a^3 - 3a^2b + 3ab^2 - b^3 + 3a^2c - 6abc + 3b^2c - 3a^2d + 6abd - 3b^2d +$$
$$3ac^2 - 6acd + 3ad^2 - 3bc^2 + 6bcd - 3bd^2 + c^3 - 3c^2d + 3cd^2 - d^3.$$

14. $(a + b + c)^2$.
15. $(x^2 + y + 1)^2$.
16. $(1 - a + b)^3$.
17. $(2 - 3x + 4y)^3$.
18. $(1 + 2x - x^2 + x^3)^2$.

EVOLUTION

109. Evolution has been defined as the operation of finding any required root of a quantity. If the quantity is not a perfect power, its root cannot be found exactly.

Thus, as 3 is not a perfect square, its square root cannot be found exactly and would be indicated by the use of the radical sign, as $\sqrt{3}$. Similarly, the cube root of 4 would be indicated, $\sqrt[3]{4}$. Such roots can be found approximately by methods shown hereafter. From the theory of exponents, we know, that,

TO FIND THE ROOT OF A SIMPLE EXPRESSION

Take the required root of the numerical coefficient and divide the exponent of each letter by the index of the required root.

From the remarks made about the algebraic signs in involution, it is clear that

1. Any even root of a positive number may be either positive or negative and hence is preceded by the sign ±. Thus, as 4 is the square of both $+2$ and -2, the square root of 4 will be ± 2.

2. A *negative number* can have *no even root,* since every

even power is positive. Thus, since -9 is not the product of two equal factors, it cannot have a square root. An indicated even root of a negative number is called an **Imaginary Number**, as $\sqrt{-4}$.

3. Any *odd root* of a number will have the *same sign* as the number itself, since every odd power of a number has the same sign as the number. Thus, since the cube of -2 is 8, the cube root of -8 must be -2, and since the cube of $+2$ is $+8$, the cube root of $+8$ must be $+2$.

To find any required root of a fraction, *take the required root of the numerator and of the denominator.*

Thus, $\sqrt[3]{\frac{8x^3}{27y^6}} = \frac{2x}{3y^2}$.

Whenever the root of a number expressed in figures is not easily found, resolve the number into its prime factors Thus, to find $\sqrt{11025}$,

$$\begin{array}{r} 5\underline{)11025} \\ 5\underline{)2205} \\ 3\underline{)441} \\ 3\underline{)147} \\ 7\underline{)49} \\ 7 \end{array}$$

Hence $\sqrt{11025} = \sqrt{3^2 \times 5^2 \times 7^2} = 3 \times 5 \times 7 = 105$

EXERCISE 73

Find the indicated root:

1. $\sqrt{9a^2b^4}$.
2. $\sqrt[3]{27x^6y^3}$.
3. $\sqrt[5]{-32a^{15}}$.
4. $\sqrt[4]{81a^8b^{12}}$.
5. $\sqrt[3]{-64y^6}$.
6. $\sqrt{64x^2y^4}$.
7. $\sqrt[3]{-125x^3}$.
8. $\sqrt[6]{64x^{12}y^{18}}$.
9. $\sqrt{4x^{4n}}$.
10. $\sqrt[3]{-a^{3n}b^{6m}}$.
11. $\sqrt{\frac{9a^2}{16b^4}}$.
12. $\sqrt[4]{\frac{625x^8}{81y^{12}}}$.
13. $\sqrt[3]{-\frac{343a^3b^3}{27x^6}}$.
14. $\sqrt[5]{-\frac{243x^{10}}{y^5}}$.
15. $\sqrt[6]{\frac{64a^{6n}}{b^{12}}}$.

SQUARE ROOTS OF COMPOUND EXPRESSIONS

110. The method for extracting the square root of a compound expression may be found by considering the square of a binomial as $(a + b)^2 = a^2 + 2ab + b^2$. The square root of $a^2 + 2ab + b^2$ is evidently $a + b$. The first term of the root is a, the square root of the first term of the expression. If a^2 is subtracted from the expression, the remainder is $2ab + b^2$.

The second term of the root b, may be found by dividing the first term of the remainder $2ab$ by $2a$, twice the root already found. This divisor is called the *trial divisor*. Since $2ab + b^2 = b(2a + b)$, if we add the second term of the root, b, to the trial divisor, we get $2a + b$, the complete divisor. Multiplying the complete divisor by the second term of the root, b, we get $2ab + b^2$. Hence,

To Extract the Square Root of a Compound Expression,

Arrange the terms of the expression according to the ascending or descending powers of the dominant letter. The square root of the first term of the expression will be the first term of the root. Subtract the first term of the expression. Double the root already found for the trial divisor. Divide the first term of the remainder by this trial divisor and the quotient will be the second term of the root. Add the new term of the root to the trial divisor and multiply this complete divisor by the new term of the root. Subtract this product from the remainder and then repeat the operation as often as necessary.

EXERCISE 74

Extract the square root of:

1. $4x^4 + 12x^3 + 5x^2 - 6x + 1$.

$$\begin{array}{r|l|l}
 & 4x^4 + 12x^3 + 5x^2 - 6x + 1 & \underline{2x^2 + 3x - 1} \\
 & 4x^4 & \\ \cline{2-2}
4x^2 + 3x & 12x^3 + 5x^2 & \\
 & 12x^3 + 9x^2 & \\ \cline{2-2}
4x^2 + 6x - 1 & -4x^2 - 6x + 1 & \\
 & -4x^2 - 6x + 1 & \\ \cline{2-2}
\end{array}$$

The expression is arranged according to the descending powers of x.

The square root of the first term $4x^4$ is $2x^2$ which is placed to the right of the expression. Subtracting $4x^4$, the remainder is $12x^3 + 5x^2 - 6x + 1$, of which only the first two terms are brought down. Doubling $2x^2$, the first term of the root, we get the trial divisor $4x^2$. Divide the first term of the remainder, $12x^3$, by the trial divisor, $4x^2$, to get $3x$, the next term of the root. Add this term to the trial divisor to get the complete divisor, $4x^2 + 3x$. Multiply the complete divisor by the new term of the root, $3x$, and subtract this product, $12x^3 + 9x^2$ from the remainder. The new remainder is $-4x^2 - 6x + 1$, bringing down the last two terms of the expression.

The new trial divisor is double the root already found, $4x^2 + 6x$. Divide the first term of the remainder by the first term of this trial divisor to get the next term of the root, -1. Complete the divisor, by adding -1 to the trial divisor and multiply the complete divisor, $4x^2 + 6x - 1$ by the new term of the root, -1. Subtract the product, $-4x^2 - 6x + 1$ and the remainder is zero. Hence the required root is $2x^2 + 3x - 1$.

2. $4x^2 - 12xy + 9y^2$. 3. $a^2x^2 - 2abcx + b^2c^2$.
4. $x^4 + 6x^2 + 4x^3 + 4x + 1$.
5. $1 + 4a^4 + 5a^2 + 12a^3 - 6a$.
6. $a^2 + b^2 + c^2 + 2ab - 2ac - 2bc$.
7. $36x^4 - 24x^3 - 8x^2 + 4x + 1$.
8. $49x^6 + 67x^4 + 9x^2 + 70x^5 + 30x^3$.
9. $x^6 - 4x^5 + 10x^4 - 14x^3 + 13x^2 - 6x + 1$.
10. $\frac{9x^2}{4a^2} + \frac{9a^2}{4x^2} + \frac{3x}{a} - \frac{7}{2} - \frac{3a}{x}$.

Arrange according to the descending powers of x, remembering that $-\frac{3a}{x}$ represents x^{-1} and should therefore follow $-\frac{7}{2}$ which contains the zero power of x.

$$\begin{array}{r|l}
 & \frac{9x^2}{4a^2}+\frac{3x}{a}-\frac{7}{2}-\frac{3a}{x}+\frac{9a^2}{4x^2} \;\Big|\; \underline{\frac{3x}{2a}+1-\frac{3a}{2x}} \\
 & \underline{\frac{9x^2}{4a^2}} \\
\frac{3x}{a}+1 & \frac{3x}{a}-\frac{7}{2} \\
 & \underline{\frac{3x}{a}+1} \\
\frac{3x}{a}+2-\frac{3a}{2x} & -\frac{9}{2}-\frac{3a}{x}+\frac{9a^2}{4x^2} \\
 & \underline{-\frac{9}{2}-\frac{3a}{x}+\frac{9a^2}{4x^2}}
\end{array}$$

11. $4x^2 + 6x + \frac{9}{4}$.

12. $\frac{x^2}{a^2} - \frac{2b}{a} + \frac{b^2}{x^2}$.

13. $\frac{x^2}{4} + \frac{9}{x^2} - x + 4 - \frac{6}{x}$.

14. $a^4 + a^3 - \frac{1}{4}a^2 - \frac{1}{4}a + \frac{1}{16}$.

15. $\frac{9x^2}{y^2} + \frac{11}{2} + \frac{y^2}{16x^2} - \frac{12x}{y} - \frac{y}{x}$.

16. $\frac{1}{9}x^2 + \frac{1}{4}y^2 - \frac{1}{3}xy + \frac{2}{15}xz - \frac{1}{5}yz + \frac{1}{25}z^2$.

17. $4x^{-4} + 12x^{-3}y^{-\frac{1}{2}} + 5x^{-2}y^{-1} - 6x^{-1}y^{-\frac{3}{2}} + y^{-2}$.

$$\begin{array}{lll}
\multicolumn{3}{l}{4x^{-4} + 12x^{-3}y^{-\frac{1}{2}} + 5x^{-2}y^{-1} - 6x^{-1}y^{-\frac{3}{2}} + y^{-2})\underline{2x^{-2} + 3x^{-1}y^{-\frac{1}{2}} - y^{-1}}} \\
\multicolumn{3}{l}{\underline{4x^{-4}}} \\
x^{-2} + 3x^{-1}y^{-\frac{1}{2}} & \big| \; 12x^{-3}y^{-\frac{1}{2}} + 5x^{-2}y^{-1} & \\
 & \big| \; \underline{12x^{-3}y^{-\frac{1}{2}} + 9x^{-2}y^{-1}} & \\
 & 4x^{-2} + 6x^{-1}y^{-\frac{1}{2}} - y^{-1} & \big| \; -4x^{-2}y^{-1} - 6x^{-1}y^{-\frac{3}{2}} + y^{-2} \\
 & & \big| \; \underline{-4x^{-2}y^{-1} - 6x^{-1}y^{-\frac{3}{2}} + y^{-2}}
\end{array}$$

18. $x^{\frac{2}{3}} + x^{\frac{1}{3}} + \frac{1}{4}$.

19. $4x^3 - 4x^{\frac{3}{2}}y^{-\frac{1}{2}} + y^{-1}$.

20. $a^{-4} + 2a^{-2}b + b^2$.

21. $x^2 - 2x + 3 - 2x^{-1} + x^{-2}$

22. $x^{\frac{1}{2}} + 4x^{\frac{1}{4}} + 2 - 4x^{-\frac{1}{4}} + x^{-\frac{1}{2}}$.

23. $x^2 + y$.

An approximate value of the square root can be found to any number of terms as follows:

$$\begin{array}{llll}
 & x^2 + y & \big| \; \underline{x + \frac{y}{2x} - \frac{y^2}{8x^3}} & \\
 & \underline{x^2} & & \\
2x + \frac{y}{2x} & \big| \; y & & \\
 & \big| \; \underline{y + \frac{y^2}{4x^2}} & & \\
2x + \frac{y}{x} - \frac{y^2}{8x^3} & & \big| \; -\frac{y^2}{4x^2} & \\
 & & \big| \; \underline{-\frac{y^2}{4x^2} - \frac{y^3}{8x^4} + \frac{y^4}{64x^6}} & \\
 & & \frac{y^3}{8x^4} - \frac{y^4}{64x^6} = \text{remainder.} &
\end{array}$$

Find the square root to three terms:

24. $a^2 + b$.

25. $x^2 - \frac{1}{2}x$.

26. $a^2 + \frac{1}{3}b$.

27. $1 + x$.

28. $4a^2 - 1$.

29. $a^2 - b^2$.

30. $1 - 2x$.

31. $9 + 6a$.

32. $4x^2 + 2y$.

ARITHMETICAL SQUARE ROOT

111. Since $1^2 = 1,$

$$10^2 = 100,$$

$$100^2 = 10000, \text{ and so on,}$$

the square root of any number between 1 and 100 must lie between 1 and 10; of any number between 100 and 10000 must lie between 10 and 100. Hence the square root of a number of one or two figures will contain one figure, the square root of a number of three of four figures will contain two figures and so on. Hence, to extract the square root of an integral number, the number is divided into groups of two figures each, beginning at the right, and the square root will contain as many figures as there are groups. If the number consists of an odd number of figures, the last group, at the left, will contain only one figure.

The rule for the square root of algebraic expressions is also used for arithmetical square root, but the student should note that, while in an algebraic expression, the position of a letter does not affect its value, this is not true of the figures that compose a number. Thus, $a + b = b + a$, but the figures of any number as 57 cannot change places without changing the value of the number, since $57 = 5$ tens $+ 7$ units, that is $= 50 + 7$.

1. Extract the square root of 1521.

```
      15'21)39
       9
    69)621
       621
```

Arrange the number in groups of two figures each. The largest square in the first group is 9 whose square root is 3. Subtracting 9, the remainder is 6. Since the square root will contain 2 figures, the 3 in reality is 30, and subtracting 9 is really the subtraction of 900, the square of 30. As 9 is placed in the hundreds' place, it is not necessary to annex the two zeros.

Doubling the root already found, we get 6 for our trial divisor. Here again, the 6 really means 60, but as the 6 is placed in the tens' place, it becomes equal to 60 as soon as the divisor is completed by annexing to the trial divisor the new figure of the root. To get this new figure of the root, divide the remainder by the trial divisor, but, as we are using only one figure of the complete divisor, one figure of the remainder is reserved; that is, we divide 62 by the trial divisor 6. The actual quotient 10 would be too large, so we take 9 as the next figure of the root. Multiplying the complete divisor 69 by the new figure of the root, we get 621, which leaves no remainder. If the number consists of more than two groups of figures, the operation is repeated, as often as necessary.

2. Extract the square root of 4,431,025.

$$\begin{array}{rl} & 4'43'10'25)\underline{2105} \\ & \underline{4} \\ 41) & 43 \\ & \underline{41} \\ 4205) & 21025 \\ & \underline{21025} \end{array}$$

In finding the third figure of the root, the trial divisor 42 is not contained in 21, the first two figures of the remainder, hence 0 is placed in the root and in the trial divisor. Then bring down the next group and proceed according to rule.

What has been said about the groups of an integral number is also true of decimals. There will be a figure of decimals in the root, for every group of two decimal figures in the number.

3. Extract the square root of .0729.

$$\begin{array}{rl} & .07'29').\underline{27} \\ & \underline{4} \\ 47) & 329 \\ & \underline{329} \end{array}$$

Arrange the number in groups of 2 figures each, beginning at the decimal point. Then proceed as before, finally pointing off as many figures of decimals, as there are groups of decimals in the number.

4. Extract the square root of 904.2049.

$$\begin{array}{rl} & 9'04.20'49')\underline{30.07} \\ & \underline{9} \\ 6007) & 042049 \\ & \underline{42049} \end{array}$$

Point off groups of two figures each, beginning at the decima point, going to the left for the whole number, and to the right for the decimal.

As there are two groups of decimals in the number, point off two decimals in the root.

EXERCISE 75

Find the square root of:

1. 324.
2. 5.76.
3. 2809.
4. 18.49.
5. 104,976.
6. 510.76.
7. 53,494,596.
8. 104.468841.
9. 1859.765625.
10. 0.000020578369.
11. 0.375.

```
        0.37'50'00'00').6123+
          36
        121)150
            121
        1222)2900
             2444
        12243)45600
              36729
               8871
```

As 0.375 is not a square, its square root is obtained approximately to as many decimals as may be required by annexing sufficient zeros to make the proper number of decimal groups.

Find the square root to 4 decimal places:

12. 0.125.
13. 11.
14. 0.5.
15. 5.
16. 1.2.
17. 2.5.
18. 0.4.
19. 0.625.
20. 0.343.
21. 0.144.

Find the square root to 4 decimal places:

22. $\frac{9}{11}$.

It is usually most convenient to change to a decimal fraction. If this division does not come out even, carry it out to the proper number of decimals, in this case, to eight places, since 4 decimal groups are required.

```
          11)9.00000000
              .81818181+
         .81'81'81'81').9045+
          81
      1804)8181
           7216
     18085)96581
           90425
            6156
```

23. $\frac{3}{8}$
24. $\frac{3}{4}$
25. $\frac{5}{6}$
26. $\frac{7}{8}$
27. $\frac{5}{7}$
28. $\frac{1}{4}$
29. $\frac{7}{2}$
30. $\frac{5}{8}$
31. $\frac{8}{9}$
32. $\frac{7}{11}$

CUBE ROOTS OF COMPOUND EXPRESSIONS

112. Since $(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$, the cube root of $a^3 + 3a^2b + 3ab^2 + b^3$ is $a + b$.

If it is required to find the cube root of $a^3 + 3a^2b + 3ab^2 + b^3$, *the first term of the root, a,* is evidently found by taking *the cube root of the first term, a^3.*

Subtracting a^3, the remainder is $3a^2b + 3ab^2 + b^3$ which equals $(3a^2 + 3ab + b^2)b$. If the first term of the remainder, $3a^2b$, is divided by $3a^2$, the second term of the root, b, is obtained. Hence *the trial divisor is three times the square of the root already found.*

The complete divisor must be $3a^2 + 3ab + b^2$. Therefore, *add to the trial divisor, $3a^2$, three times the product of the first and second terms of the root, $3ab$, and also the square of the second term of the root, b^2.*

In longer expressions, the same method is used, considering, a, the part of the root already found and, b, the new term of the root. If we distinguish between a and b, by calling a, the *old* and b the *new* part of the root, the rule for cube root may be briefly expressed:

Arrange the terms of the expression according to the ascending or descending powers of the dominant letter.

The cube root of the first term of the expression is the first term of the cube root.

Subtract the cube of the first term of the root.

The trial divisor is three times the square of the "old."

Divide the first term of the remainder by the first term of the trial divisor to get the next term of the root. To complete the divisor, add to the trial divisor, three times the product of the "old" by the "new" and the square of the "new."

Subtract the product of the complete divisor by the "new" term of the root and if there is a remainder repeat the operation.

Find the cube root of $12a^5b - 13a^3b^3 + 27ab^5 + 8a^6 + 45a^2b^4 - 30a^4b^2 - 27b^6$.

$$\begin{array}{r|l}
 & \hfill \underline{\left|2a^2 + ab - 3b^2\right.} \\
 & 8a^6 + 12a^5b - 30a^4b^2 - 35a^3b^3 + 45a^2b^4 + 27ab^5 - 27b^6 \\
 & 8a^6 \\
\hline
12a^4 \qquad\qquad & 12a^5b - 30a^4b^2 - 35a^3b^3 \\
6a^3b \qquad & \\
a^2b^2 & \\
\hline
12a^4 + 6a^3b + a^2b^2 & 12a^5b + 6a^4b^2 + a^3b^3 \\
\hline
12a^4 + 12a^3b + 3a^2b^2 \qquad\qquad & -36a^4b^2 - 36a^3b^3 + 45a^2b^4 + 27ab^5 - 27b^6 \\
-18a^2b^2 - 9ab^3 \qquad & \\
+9b^4 & \\
\hline
12a^4 + 12a^3b - 15a^2b^2 - 9ab^3 + 9b^4 & -36a^4b^2 - 36a^3b^3 + 45a^2b^4 + 27ab^5 - 27b^6 \\
\hline
\end{array}$$

First trial divisor is $3 \times (2a^2)^2 = 12a^4$

To complete the divisor, add, $3 \times 2a^2 \times ab = 6a^3b$ and $(ab)^2 = a^2b^2$

Second trial divisor is $3(2a^2 + ab)^2 = 12a^4 + 12a^3b + 3a^2b^2$

To complete, add, $3(2a^2 + ab)(-3b^2) = -18^2b^2 - 9ab^3$ and $(-3b^2)^2 = +9b^4$

The rule for the divisor may be easily memorized in this form:

Trial Divisor = 3 old²

to complete Divisor add 3 old × new + new²

EXERCISE 76

Extract the cube root of:

1. $x^3 + 3x^2y + 3xy^2 + y^3$.
2. $8a^3 - 12a^2b + 6ab^2 - b^3$.
3. $27y^3 - 8 + 36y - 54y^2$.
4. $a^6 - 3a^5b + 5a^3b^3 - 3ab^5 - b^6$.
5. $x^6 + 2a^2x^4 + 6a^4x^2 - 3ax^5 - 7a^3x^3 - 3a^5x + a^6$.
6. $x^3 - \dfrac{3x^2y}{2} + \dfrac{3xy^2}{4} - \dfrac{y^3}{8}$.
7. $\dfrac{a^6}{b^9} - \dfrac{3a}{b^4} + \dfrac{3b}{a^4} - \dfrac{b^6}{a^9}$.
8. $8a^{-3} - 12a^{-2}b^{-2} + 6a^{-1}b^{-4} - b^{-6}$.
9. $x^{\frac{3}{2}} + 9xy^{\frac{1}{2}} + 27x^{\frac{1}{2}}y^{\frac{2}{3}} + 27y$.
10. $8a - 36a^{\frac{2}{3}}y^{-\frac{1}{2}} + 54a^{\frac{1}{3}}y^{-1} - 27y^{-\frac{3}{2}}$.

ARITHMETICAL CUBE ROOT

113. Since, $1^3 = 1$, $10^3 = 1000$, $100^3 = 1{,}000{,}000$ and so on, the cube root of any number between 1 and 1000, that is of a number of one, two, or three figures, will contain one figure. The cube root of any number between 1000 and 1,000,000, that is, of a number consisting of four, five, or six figures, will contain two figures, and so on.

Therefore, if any number is divided into groups of three figures each, its cube root will contain one figure for each group. The last group, to the left, may be incomplete, containing one, two, or three figures.

Since in the multiplication of decimals, the number of decimal places in the product is equal to the sum of the number of places in multiplicand and multiplier, the cube of any decimal will contain three times as many places as the number. Thus $(.12)^3 = .001728$. Therefore, if any decimal fraction is divided into groups of three figures each, its cube root will contain as many places as there are groups of decimals. The last group, to the right, must also contain three figures, zeros being added if necessary. The rule for finding the cube root of algebraic expressions is also used in finding arithmetical cube roots.

Find the cube root of 74.246873427.

Separate the number into groups of three figures each, *beginning at the decimal point.*

$$
\begin{array}{rcr|l}
\multicolumn{3}{r}{} & 74.246'873'427\ )\underline{4.203} \\
\multicolumn{3}{r}{} & 64 \\
\hline
3 \times 40^2 & = & 4{,}800 & 10{,}246 \\
3 \times 40 \times 2 & = & 240 & \\
2^2 & = & 4 & \\
\cline{3-3}
 & & 5{,}044 & 10{,}088 \\
\hline
3 \times 420^2 & = & 529{,}200 & 158{,}873{,}427 \\
\hline
3 \times 4200^2 & = & 52{,}920{,}000 & \\
3 \times 4200 \times 3 & = & 37{,}800 & \\
3^2 & = & 9 & \\
\cline{3-3}
 & & 52{,}957{,}809 & 158{,}873{,}427 \\
\hline
\end{array}
$$

As in arithmetical square roots, the student should note that the position of a figure gives it its value.

Thus, in finding the second figure of the root, the first figure, 4, has the value of 4 tens, that is, 40 as compared to the second figure of the root, 2. Hence in finding the trial divisor, three times the square of the old is three times the square of 40.

In finding the third figure of the root, the trial divisor, 3 times the square of 420 or 529,200 is not contained in the remainder 158,873.

Therefore, a zero is placed in the root and the trial divisor becomes 3 times the square of 4200. The next group 427 is brought down and the remainder divided by the trial divisor.

Since the number contained three groups of decimals, the cube root contains three decimals.

EXERCISE 77

Find the cube root of:

1. 12167.
2. 33,076,161.
3. 66,430,125.
4. 40001.688.
5. 128.024064.
6. 1740.992427.

The cube roots of numbers that are not exact cubes, may be found approximately to any number of decimal places by annexing a sufficient number of zeros to supply the proper number of decimal groups.

Find, to four decimal places, the cube root of:

7. 23.
8. 111.
9. 2.7.
10. 1.25
11. $\frac{1}{2}$
12. $\frac{2}{3}$
13. $\frac{3}{4}$
14. $\frac{7}{11}$

ROOTS THAT MAY BE RESOLVED INTO SQUARE ROOTS OR CUBE ROOTS

114. Since the fourth power is the square of a square, the sixth power, the square of the cube, the eight power, the square of the square of the square and so on, the fourth root will be the square root of the square root, the sixth root, the cube root of the square root and so on.

The index of the root should be resolved into its factors, 2 and 3, and the operation performed accordingly. Thus as $12 = 2 \times 2 \times 3$, the 12th root is found by taking the cube root of the square root of the square root.

It is advisable to take the square root before the cube root as it is less complex.

EXERCISE 78

Find the fourth root of:

1. $a^4 - 4a^3b + 6a^2b^2 - 4ab^3 + b^4$.
2. $16x^4 - 96x^3y + 216x^2y^2 - 216xy^3 + 81y^4$.

3. $1 + 10x^2 - 4x + 19x^4 - 16x^5 - 16x^3 + x^8 - 4x^7 + 10x^6$.

Find the sixth root of:

4. $60a^4b^2 + 12a^5b + a^6 + 64b^6 + 160a^3b^3 + 240a^2b^4 + 192b^5$.

5. $729x^6 - 2916x^5y + 4860x^4y^2 - 2160x^3y^3 + 2160x^2y^4 - 576xy^5 + 64y^6$.

Find the eight root of:

6. $256 + 1024x + 1792x^2 + 1792x^3 + 1120x^4 + 448x^5 + 112x^6 + 16x^7 + x^8$.

REVIEW EXERCISE XIV

Expand:

1. $(2a^2 - b^3)^5$.
2. $(2b^2 - 3y)^4$.
3. $(3x - 2a)^5$.
4. $(3x + 2y)^4$.
5. $(1 - 3x^2)^4$.
6. $(a^2 - 2x^3)^6$.
7. $(2a^2 - 3b^3)^4$.
8. $(4b^2 - 3x)^4$.
9. $(2a - b^2)^6$.
10. $\left(1 - \frac{x}{2}\right)^5$

Find the square root of:

11. $1 + 9x$ to three terms.
12. $x^8 + 3x^4 + 1 - 2x^6 - 2x^2$.
13. .97 to three decimals.
14. $1 + x$ to four terms.
15. $34x^3 - 22x^4 + x^6 + 289 - 374x + 121x^2$.
16. $p^2 - s$ to four terms.
17. 104976.
18. $x^4 + x^3 + \frac{13x^2}{20} + \frac{x}{5} + \frac{1}{25}$.
19. $\frac{a^4}{4} + a^3x + \frac{4a^2x^2}{3} + \frac{2ax^3}{3} + \frac{x^4}{9}$
20. $R^2 - \frac{3A}{2}$ to three terms.
21. $\frac{2}{3}$ to four decimals.
22. $9a - 12a^{\frac{1}{2}} + 10 - 4a^{-\frac{1}{2}} + a^{-1}$.

Find the cube root of:

23. 87 to two decimals.

24. .7 to three decimals.

25. $1 + 6d + d^6 + 6d^5 + 15d^4 + 20d^3 + 15d^2$.

26. $8x^6 + 1 - 9x - 63x^3 + 33x^2 + 66x^4 - 36x^5$.

Find the fourth root of

27. $a^6 - 4a^4\sqrt{ab^{-1}} + 6a^3b^{-1} - 4ab^{-1}\sqrt{ab^{-1}} + b^{-2}$

CHAPTER XV

RADICALS

115. Any indicated root of a number or algebraic expression is a **radical expression** or simply a **radical.**

As $\sqrt[3]{8}$, $\sqrt{a}$, $\sqrt[5]{a^2+b^2}$.

The quantity under the radical sign is the **radicand.**

An indicated root that cannot be found exactly is called a **surd.**

As $\sqrt{2}$, $\sqrt[3]{4}$, $\sqrt[5]{x+y}$.

The **order** of the surd is shown by the index.

Thus, $\sqrt{2}$ is a surd of the *second order* or a *quadratic surd.*
$\sqrt[3]{4}$ is a surd of the *third order* or a *cubic surd.*
$\sqrt[4]{x}$ is a surd of the *fourth order* or a *biquadratic surd*, and so on.

An expression containing both rational and radical factors is called a **mixed surd,** as $5\sqrt{2}$, $x\sqrt[3]{y}$, and the rational factor is called the *coefficient* of the surd. When a surd has no rational coefficient except 1, it is called an **entire surd.** As $\sqrt{5}$, $\sqrt[3]{x^2-y^2}$.

A surd is in its *simplest form* when the quantity under the radical sign is *integral and as small as possible.*

Surds are **similar** if they have the same surd factor when reduced to their simplest forms.

As, $\sqrt{3}$, $4\sqrt{3}$, $\frac{1}{2}\sqrt{3}$.

REDUCTION OF RADICALS

Case I

116. When the radical is a perfect power whose exponent is a factor of the index of the root.

Note that the word *simplify* still means *to perform the indicated operations.*

Thus in simplifying a radical of Case I, the indicated operation is performed as far as possible.

Thus, $\sqrt[4]{x^2}$ can be performed only partially as x^2 is not a fourth power; but, since the fourth root is the square root of the square root, one of these square roots can be extracted. This is most easily done by using fractional exponents as explained under the Theory of Exponents.

Thus $\sqrt[4]{x^2} = x^{\frac{2}{4}} = x^{\frac{1}{2}} = \sqrt{x}$.

Similarly $\sqrt[6]{\frac{8x^3y^6}{27a^3b^3}} = \left(\frac{2^3x^3y^6}{3^3a^3b^3}\right)^{\frac{1}{6}} = \frac{2^{\frac{3}{6}}x^{\frac{3}{6}}y^{\frac{6}{6}}}{3^{\frac{3}{6}}a^{\frac{3}{6}}b^{\frac{3}{6}}} = \frac{2^{\frac{1}{2}}x^{\frac{1}{2}}y}{3^{\frac{1}{2}}a^{\frac{1}{2}}b^{\frac{1}{2}}} = y\sqrt{\frac{2x}{3ab}}$.

Hence, the following rule:

Divide the exponent of the power by the index of the root.

Note.—In problems involving radicals, arithmetical numbers should be resolved into their prime factors.

EXERCISE 79

Simplify:

1. $\sqrt[4]{9}$.
2 $\sqrt[8]{16}$.
3. $\sqrt[6]{27}$.
4. $\sqrt[10]{32}$.
5. $\sqrt[4]{x^2y^2}$.
6. $\sqrt[6]{a^2b^2}$.
7. $\sqrt[8]{81x^4y^4}$.
8. $\sqrt[6]{8a^3b^6}$.
9. $\sqrt[4]{\frac{4x^2}{25y^2}}$.
10. $\sqrt[6]{\frac{9a^2b^2}{25(x-a)^2}}$.
11. $\sqrt[8]{\frac{16x^4y^8}{a^4b^8}}$.

Case II

117. When the radical contains a factor which is a perfect power of the same degree as the radical.

1. $\sqrt{x^2y} = \sqrt{x^2} \times \sqrt{y} = x\sqrt{y}$.
2. $\sqrt{a^2b^3} = \sqrt{a^2b^2} \times \sqrt{b} = ab\sqrt{b}$.
3. $\sqrt[3]{500} = \sqrt[3]{125 \times 4} = \sqrt[3]{125} \times \sqrt[3]{4} = 5\sqrt[3]{4}$.
4. $3\sqrt{50x^3y^3} = 3\sqrt{25x^2y^2 \times 2xy} = 3\sqrt{25x^2y^2} \times \sqrt{2xy} =$
$3 \times 5xy\sqrt{2xy} = 15xy\sqrt{2xy}$.
5. $2\sqrt[3]{\frac{(x-y)^4}{8x^2}} = 2\sqrt[3]{\frac{(x-y)^3}{2^3}} \times \sqrt[3]{\frac{x-y}{x^2}} = 2 \times \frac{x-y}{2}\sqrt[3]{\frac{x-y}{x^2}}$
$= (x-y)\sqrt[3]{\frac{x-y}{x^2}}$.

The indicated operation is performed as far as possible by extracting the indicated root of the factor whose root can be extracted.

Hence, the following rule:

Resolve the radical into two factors, one of which is the greatest perfect power of the same degree as the radical. Ex-

tract the indicated root of this factor which then becomes the rational coefficient of the resulting surd.

EXERCISE 80

Simplify:

1. $\sqrt{18}$.
2. $\sqrt{48}$.
3. $\sqrt{2000}$.
4. $\sqrt[3]{54}$.
5. $\sqrt[3]{250}$.
6. $\sqrt[3]{270}$.
7. $\sqrt[4]{48}$.
8. $\sqrt[4]{243}$.
9. $\sqrt[5]{64}$.
10. $\sqrt[5]{486}$.
11. $3\sqrt{243}$.
12. $2\sqrt[3]{135}$.
13. $5\sqrt[4]{810}$.
14. $4\sqrt{m^3n}$.
15. $3\sqrt[3]{8a^2b^6}$.
16. $7\sqrt[4]{16a^5b^5}$.
17. $5\sqrt[5]{243x^6y^{12}}$.
18. $a\sqrt[3]{1372a^7}$.
19. $\sqrt{\dfrac{32a^3b^3}{45x^2y^2}}$.
20. $\dfrac{1}{2}\sqrt[3]{\dfrac{8a^2}{b^2}}$.
21. $3\sqrt[4]{\dfrac{16x^6y^7}{81a^4b^4}}$.
22. $\dfrac{2ab}{3}\sqrt[5]{\dfrac{243x^6}{32y}}$.
23. $2\sqrt{(a-b)^3}$.
24. $\sqrt[3]{\dfrac{(2x -- 3y)^4}{8}}$.
25. $\dfrac{a}{b}\sqrt[4]{\dfrac{b^5(x+y)^6}{a^5(x-y)^6}}$.
26. $\dfrac{2xy}{3c}\sqrt[3]{\dfrac{54c^4}{8x^5y^5}}$.

Note.—The easiest method of resolving a number into its prime factors is the method usually taught in arithmetic.

Thus,

$$\begin{array}{r} 2\underline{)1250} \\ 5\underline{)625} \\ 5\underline{)125} \\ 5\underline{)25} \\ 5 \end{array}$$

1250 contains one 2 and four 5's, hence,

$$1250 = 2 \times 5^4$$

Case III

118. When the radical is a fraction whose denominator is not a perfect power of the same degree as the radical.

A surd is not in its simplest form unless the quantity under the radical sign is integral. If the denominator is not a perfect power of the same degree as the radical, the indicated root of this denominator cannot be extracted. The value of the fraction is not changed if numerator and denominator are multiplied by the same factor. Hence, the denominator can be made a perfect power by multiplying both terms of the fraction by the necessary *rationalizing factor.*

Thus, $\sqrt[3]{\frac{3}{4}} = \sqrt[3]{\frac{3}{4}\times\frac{2}{2}} = \sqrt[3]{\frac{6}{8}} = \sqrt[3]{\frac{1}{8}\times 6} = \frac{1}{2}\sqrt[3]{6}$.
And $5\sqrt[5]{\frac{5}{81}} = 5\sqrt[5]{\frac{5}{3^4}\times\frac{3}{3}} = 5\sqrt[5]{\frac{1}{3^5}\times 15} = \frac{5}{3}\sqrt[5]{15}$.

Hence, the following rule:

Multiply both terms of the fraction by a factor that will make the denominator a perfect power of the same degree as the radical.

Then proceed as in Case II.

EXERCISE 81

Simplify:

1. $\sqrt{\frac{1}{2}}$.
2. $\sqrt[3]{\frac{2}{3}}$.
3. $\sqrt[4]{\frac{5}{8}}$.
4. $\frac{1}{2}\sqrt{\frac{1}{3}}$.
5. $\frac{2}{3}\sqrt{\frac{3}{8}}$.
6. $\frac{1}{2}\sqrt[3]{\frac{8}{9}}$.
7. $\frac{3}{4}\sqrt[4]{\frac{7}{80}}$.
8. $\frac{2}{3}\sqrt[5]{\frac{9}{16}}$.
9. $\frac{5}{8}\sqrt[3]{\frac{40}{1715}}$.
10. $\sqrt{\frac{x}{y}}$.
11. $2\sqrt{\frac{a^2b}{cx^2}}$.
12. $\sqrt[3]{\frac{25x}{27y}}$.
13. $\frac{a}{b}\sqrt{\frac{3b^3}{8a^3}}$.
14. $xy\sqrt[3]{\frac{3ab}{16x^2y^2}}$.
15. $\frac{2x}{3y}\sqrt[4]{\frac{81y^5}{8x^3}}$.
16. $\sqrt{\frac{1}{(x-y)^3}}$

TO CHANGE A MIXED SURD TO AN ENTIRE SURD

119. This operation is the reverse of simplifying a radical. Instead of extracting the indicated root of a factor and taking it outside the radical sign, the factor, which is called the coefficient, is to be placed under the radical sign. Since the value of the expression must be unchanged, the factor must be raised to a power of the same degree as the radical.

Thus, $2\sqrt[3]{5} = \sqrt[3]{2^3}\times\sqrt[3]{5} = \sqrt[3]{2^3\times 5} = \sqrt[3]{40}$.
Also, $2a\sqrt[4]{3b} = \sqrt[4]{(2a)^4}\times\sqrt[4]{3b} = \sqrt[4]{16a^4\times 3b} = \sqrt[4]{48a^4b}$.

Hence, the following rule:

Raise the coefficient to a power of the same degree as the radical and place this power under the radical sign as a surd factor.

EXERCISE 82

Change to entire surds:

1. $3\sqrt{3}$.
2. $2\sqrt{7}$.
3. $2\sqrt[3]{3}$.
4. $3\sqrt[4]{2}$.
5. $2\sqrt[5]{5}$.
6. $\frac{1}{2}\sqrt{2}$.
7. $-\frac{1}{3}\sqrt[3]{3}$.
8. $-2\sqrt[5]{y}$.
9. $-\frac{1}{2}\sqrt[4]{x^3}$.
10. $-\frac{2}{3}\sqrt[3]{\frac{9ab}{2}}$.
11. $\frac{5}{8}\sqrt{\frac{2}{5}}$.
12. $\frac{m}{n}\sqrt[3]{\frac{n^5}{m^5}}$.
13. $\frac{2x}{3y}\sqrt{\frac{27y}{8x}}$.
14. $(a-b)\sqrt[3]{(a+b)^2}$.

REDUCTION OF RADICALS TO A COMMON INDEX

120. If radicals are expressed by the use of fractional exponents, the denominator represents the index of the root. If the fractional exponents are changed to fractions having a common denominator, the common denominator will represent the common index of the radicals.

1. Reduce to a common index,

$$\sqrt{3} \text{ and } \sqrt[3]{2}.$$

Using fractional exponents,

$$3^{\frac{1}{2}} \text{ and } 2^{\frac{1}{3}}$$

Change to a common denominator,

$$3^{\frac{3}{6}} \text{ and } 2^{\frac{2}{6}}$$

Write with radical signs,

$$\sqrt[6]{3^3} \text{ and } \sqrt[6]{2^2}.$$

Which equal, $\sqrt[6]{27}$ and $\sqrt[6]{4}$.

Hence, the following rule:

Write the radicals with fractional exponents. Change these to equivalent fractional exponents having the least common denominator. Write with radical signs, using the common denominator as the common index and the numerators as the exponents of the respective radicals.

EXERCISE 83

Reduce to surds of the same order:

1. $\sqrt{2}$ and $\sqrt[3]{3}$.
2. $\sqrt[3]{2}$ and $\sqrt[4]{3}$.
3. $\sqrt{5}$ and $\sqrt[4]{10}$.
4. $\sqrt[3]{9}$ and $\sqrt[6]{47}$.
5. $\sqrt{ab}$ and $\sqrt[3]{xy}$.
6. $\sqrt{3}$, $\sqrt[3]{2}$ and $\sqrt[4]{5}$.
7. $\sqrt[3]{a^2}$, $\sqrt[5]{a^3}$ and $\sqrt[10]{a^4}$.
8. $\sqrt[4]{a-b}$ and $\sqrt[6]{a+b}$.

If surds of different orders are reduced to a common index, they can be compared as to magnitude.

Arrange in descending order of magnitude.

9. $\sqrt{2}$, $\sqrt[3]{3}$ and $\sqrt[4]{4}$.
10. $\sqrt[3]{7}$, $\sqrt[6]{50}$ and $\sqrt{5}$.
11. $\sqrt{3}$, $\sqrt[6]{13}$ and $\sqrt[10]{150}$.
12. $\sqrt[3]{5}$, $\sqrt[4]{7}$ and $\sqrt[6]{15}$.

ADDITION AND SUBTRACTION OF RADICALS

121. In algebraic addition, like terms only can be combined by algebraically adding their coefficients. Thus, $2x^2 + 3x^2 - 4x^2$ can be added and their sum written as one term $(2 + 3 - 4)x^2 = x^2$, but $2x$ and $3y$ can only be added by connecting them with the proper sign, as, $2x + 3y$, which is an indicated addition. In the same way, similar surds only can be combined in algebraic addition and are treated just as like terms. Thus $3\sqrt{3} + 2\sqrt{3} - 4\sqrt{3} = (3 + 2 - 4)\sqrt{3} = \sqrt{3}$ but $3\sqrt{2}$ and $2\sqrt{3}$ can only be added by connecting them with the + sign, as, $3\sqrt{2} + 2\sqrt{3}$.

Hence, to add radicals, follow the rule:

Reduce all surds to their simplest form. Add the coefficients of similar surds and to this sum annex the common surd factor. Dissimilar surds are connected by their proper signs.

EXERCISE 84

1. Simplify $\sqrt{50} + \sqrt{32} - \sqrt{18}$.

$\sqrt{50} + \sqrt{32} - \sqrt{18} =$
$\sqrt{25 \times 2} + \sqrt{16 \times 2} - \sqrt{9 \times 2} =$
$5\sqrt{2} + 4\sqrt{2} - 3\sqrt{2} =$
$(5 + 4 - 3)\sqrt{2} = 6\sqrt{2}$.

2. Simplify $2\sqrt{\frac{2}{3}} - 3\sqrt{\frac{1}{6}} + 5\sqrt{\frac{3}{2}}$.

$2\sqrt{\frac{2}{3}} - 3\sqrt{\frac{1}{6}} + 5\sqrt{\frac{3}{2}} = 2\sqrt{\frac{6}{9}} - 3\sqrt{\frac{6}{36}} + 5\sqrt{\frac{6}{4}} =$
$2\sqrt{\frac{1}{9} \times 6} - 3\sqrt{\frac{1}{36} \times 6} + 5\sqrt{\frac{1}{4} \times 6} =$
$2 \times \frac{1}{3}\sqrt{6} - 3 \times \frac{1}{6}\sqrt{6} + 5 \times \frac{1}{2}\sqrt{6} =$
$\frac{2}{3}\sqrt{6} - \frac{1}{2}\sqrt{6} + {}^{5}\sqrt{6} =$
$(\frac{2}{3} - \frac{1}{2} + \frac{5}{2})\sqrt{6} =$
$(\frac{4}{6} - \frac{3}{6} + \frac{15}{6})\sqrt{6} = \frac{8}{3}\sqrt{6}$.

3. Simplify $2\sqrt[3]{40} - \sqrt[3]{\frac{5}{8}}$.

$2\sqrt[3]{40} - \sqrt[3]{\frac{5}{8}} =$
$2\sqrt[3]{8 \times 5} - \sqrt[3]{\frac{1}{8} \times 5} =$
$2 \times 2\sqrt[3]{5} - \frac{1}{2}\sqrt[3]{5} =$
$4\sqrt[3]{5} - \frac{1}{2}\sqrt[3]{5} = \frac{7}{2}\sqrt[3]{5}$.

Simplify:

4. $3\sqrt{7} - 5\sqrt{7} + 9\sqrt{7}$.
5. $2\sqrt[3]{4} + 3\sqrt[3]{4} - 7\sqrt[3]{4}$.
6. $\sqrt{8} - \sqrt{18} + \sqrt{32}$.
7. $\sqrt[3]{24} + \sqrt[3]{81} - \sqrt[3]{192}$.
8. $2\sqrt{48} - \frac{1}{3}\sqrt{27} - \frac{1}{5}\sqrt{75}$.
9. $\frac{1}{2}\sqrt[3]{3} + 1\frac{1}{2}\sqrt[3]{24} - \frac{1}{3}\sqrt[3]{81}$.
10. $5\sqrt[4]{2} - 2\sqrt[4]{32} + \sqrt[4]{162}$.
11. $\sqrt{\frac{1}{2}} + \sqrt{\frac{2}{25}} - \sqrt{\frac{8}{9}}$.
12. $3\sqrt{x} + \frac{1}{2}\sqrt{x} - \frac{2}{3}\sqrt{x}$.
13. $\sqrt[3]{x^2} - \frac{1}{2}\sqrt[3]{8x^2} + \frac{3}{4}\sqrt[3]{64x^2}$.
14. $a\sqrt{x} - \sqrt{b^2x} + \sqrt{a^2x}$.
15. $2\sqrt[3]{x^3} + \sqrt[3]{8x^2} - \sqrt[3]{x^2}$.
16. $3\sqrt{20xy} - 2\sqrt{45xy} - \sqrt{4xy}$.
17. $7\sqrt{49} + \sqrt{28} - 3\sqrt{63} - \sqrt{25}$.
18. $4\sqrt{\frac{1}{2}} - \sqrt[3]{81} + \frac{1}{2}\sqrt{8} + \sqrt[3]{375}$.
19. $3\sqrt[3]{m^4n} - 5\sqrt[3]{8n} + \sqrt[3]{n^4} + \frac{1}{2}\sqrt[3]{27n^7}$.

MULTIPLICATION OF RADICALS

122. In simplifying radicals, it was shown that $\sqrt{ab} = \sqrt{a} \times \sqrt{b}$, hence $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$. This is true of any radicals having a common index.

Thus $2\sqrt{27} \times 3\sqrt{18} =$
$2\sqrt{9 \times 3} \times 3\sqrt{9 \times 2} =$
$6\sqrt{3} \times 9\sqrt{2} = 6 \times 9\sqrt{3 \times 2} = 54\sqrt{6}$

If the radicals do not have a common index, they must be expressed with a common index before multiplying.

Thus, $2\sqrt{3} \times \sqrt[3]{4} = 2 \times 3^{\frac{1}{2}} \times 4^{\frac{1}{3}} = 2 \times 3^{\frac{3}{6}} \times 4^{\frac{2}{6}} =$
$2\sqrt[6]{27} \times \sqrt[6]{16} = 6\sqrt[6]{27 \times 16} = 6\sqrt[6]{432}$

Hence, the following rule:

Reduce the radicals to a common index. Multiply the *coefficients for the required coefficient and the surd factors for the required surd factor. Then simplify the result.*

EXERCISE 85

Find the product of:

1. $\sqrt{8} \times \sqrt{18}$.
2. $\sqrt{3} \times \sqrt{12}$.
3. $\sqrt{125} \times \sqrt{75}$.
4. $\sqrt[3]{5} \times \sqrt[3]{25}$.
5. $\sqrt[4]{8} \times 2\sqrt[4]{2}$.
6. $\sqrt[5]{27} \times \sqrt[5]{9}$.
7. $\sqrt[3]{4} \times \sqrt[3]{16}$.
8. $\frac{1}{2}\sqrt{7} \times \frac{2}{3}\sqrt{21}$.
9. $\sqrt{\frac{1}{2}} \times \sqrt{\frac{2}{3}}$.
10. $(\sqrt{3} + \sqrt{6}) \times \sqrt{2}$.
11. $(\sqrt[3]{16} - 2\sqrt[3]{81} + \sqrt[3]{\frac{1}{2}}) \times \sqrt[3]{9}$.
12. $\sqrt{3} \times \sqrt[3]{2}$.
13. $\sqrt[3]{\frac{1}{2}} \times \sqrt[4]{\frac{1}{3}}$.
14. $\sqrt{2} \times \sqrt[4]{8}$.
15. $\sqrt[3]{16} \times \sqrt[6]{32}$.
16. $\sqrt{\frac{2}{9}} \times \sqrt[3]{\frac{1}{3}}$.
17. $\sqrt[3]{\frac{2}{3}} \times \sqrt[4]{\frac{1}{2}}$.
18. $\sqrt[6]{81} \times \sqrt{\frac{1}{3}}$.
19. $\sqrt[5]{16} \times \sqrt{\frac{1}{2}}$.
20. $\sqrt[3]{x^2} \times \sqrt{x^3}$.

123. The usual method for multiplying polynomials is followed in the case of compound radicals.

1. Multiply $3\sqrt{2} + 2\sqrt{ab}$ by $2\sqrt{2} - 3\sqrt{ab}$.

$$\begin{array}{rrr}
3\sqrt{2} & + 2\sqrt{ab} & \\
2\sqrt{2} & - 3\sqrt{ab} & \\
\hline
12 & + 4\sqrt{2ab} & \\
 & - 9\sqrt{2ab} & - 6ab \\
\hline
12 & - 5\sqrt{2ab} & - 6ab
\end{array}$$

2. Multiply $\sqrt{a + \sqrt{b}}$ by $\sqrt{a - \sqrt{b}}$.

$$\sqrt{a + \sqrt{b}} \times \sqrt{a - \sqrt{b}} = \sqrt{(a + \sqrt{b})(a - \sqrt{b})} = \sqrt{a^2 - b}.$$

EXERCISE 86

Multiply:

1. $2\sqrt{3} + 3\sqrt{2}$ by $3 - \sqrt{3}$.
2. $\sqrt{27} - \sqrt{50}$ by $\sqrt{8} + \sqrt{5}$.
3. $\sqrt[3]{3} + \sqrt[3]{4}$ by $\sqrt[3]{9} - \sqrt[3]{2}$.
4. $\sqrt[4]{8} - \sqrt[4]{27}$ by $\sqrt[4]{3} - \sqrt[4]{2}$.
5. $3\sqrt{2} + 2\sqrt{x}$ by $2\sqrt{2} + 4\sqrt{x}$.

6. $\sqrt{6 + \sqrt{11}}$ by $\sqrt{6 - \sqrt{11}}$.
7. $2\sqrt{5 + \sqrt{4}}$ by $\sqrt{5 - \sqrt{4}}$.
8. $3\sqrt{5} + 4\sqrt{3} - 3\sqrt{2}$ by $2\sqrt{5} - 3\sqrt{6}$.
9. $\frac{1}{2}\sqrt{3} - \frac{2}{3}\sqrt{2} + \frac{3}{4}\sqrt{5}$ by $\frac{1}{3}\sqrt{3} + \frac{1}{4}\sqrt{5}$.
10. $5\sqrt{6} + 4\sqrt{5} - 2\sqrt{3}$ by $3\sqrt{6} - 2\sqrt{5} + 3\sqrt{3}$.
11. $\sqrt[3]{4} + 2\sqrt[3]{6} + 3\sqrt[3]{16}$ by $\sqrt[3]{4} + \sqrt[3]{6} + \sqrt[3]{2}$.
12. $2\sqrt{\frac{2}{3}} - \frac{1}{2}\sqrt{\frac{3}{4}} - \frac{1}{3}\sqrt{\frac{1}{6}}$ by $4\sqrt{\frac{1}{3}} + 3\sqrt{\frac{1}{2}} - \frac{1}{2}\sqrt{\frac{1}{6}}$.
13. $\frac{1}{2}\sqrt{27} + \frac{1}{3}\sqrt{32} - \frac{2}{3}\sqrt{54}$ by $\sqrt{3} - \sqrt{2} + \sqrt{8}$.
14. $2\sqrt{12} + \sqrt{\frac{1}{2}} - \sqrt{\frac{2}{3}}$ by $3\sqrt{27} - \frac{1}{3}\sqrt{8} + \sqrt{\frac{1}{6}}$.
15. $\frac{1}{2}\sqrt[3]{\frac{1}{2}} + \frac{1}{3}\sqrt[3]{\frac{1}{3}} - \frac{1}{4}\sqrt[3]{\frac{1}{4}}$ by $\frac{1}{3}\sqrt[3]{\frac{1}{2}} - \frac{1}{2}\sqrt[3]{\frac{1}{3}} + \frac{1}{5}\sqrt[3]{\frac{1}{4}}$.

DIVISION OF RADICALS

124. If the division of $\sqrt{a}$ by $\sqrt[3]{a}$ is performed by means of the exponents, that is $a^{\frac{1}{2}} \div a^{\frac{1}{3}}$, and the exponent of the divisor, $\frac{1}{3}$, is subtracted from the exponent of the dividend, $\frac{1}{2}$, the fractional exponents must be reduced to a common denominator.

Thus $a^{\frac{1}{2}} \div a^{\frac{1}{3}} = a^{\frac{3}{6} - \frac{2}{6}} = a^{\frac{1}{6}}$.

Hence, to divide radicals, they must be reduced to a common index, the denominator of the fractional exponent representing the index of the root.

Also since, $a^{\frac{3}{6}} = \sqrt[6]{a^3} = \sqrt[6]{a^2} \times \sqrt[6]{a}$.

Therefore, $\dfrac{\sqrt[6]{a^3}}{\sqrt[6]{a^2}} = \dfrac{\sqrt[6]{a^2} \times \sqrt[6]{a}}{\sqrt[6]{a^2}} = \sqrt[6]{a}$.

Hence the rule for division of radicals,

If necessary, express the radicals with a common index. Divide the coefficient of the dividend by the coefficient of the divisor for the required coefficient. Divide the surd factor of the dividend by the surd factor of the divisor for the required surd factor. Simplify the result.

1. Divide $6\sqrt{18}$ by $2\sqrt{8}$.

$$\frac{6\sqrt{18}}{2\sqrt{8}} = 3\sqrt{\frac{18}{8}} = 3\sqrt{\frac{9}{4}} = 3 \times \frac{3}{2} = \frac{9}{2}.$$

2. Divide $2\sqrt{3}$ by $3\sqrt[3]{4}$.

$$\frac{2\sqrt{3}}{3\sqrt[3]{4}}=\frac{2\cdot 3^{\frac{1}{2}}}{3\cdot 4^{\frac{1}{3}}}=\frac{2\cdot 3^{\frac{3}{6}}}{3\cdot 4^{\frac{2}{6}}}=\frac{2\sqrt[6]{3^3}}{3\sqrt[6]{4^2}}=\frac{2}{3}\sqrt[6]{\frac{3^3}{2^4}}=\frac{2}{3}\sqrt[6]{\frac{3^3}{2^4}\times\frac{2^2}{2^2}}=\frac{2}{3}\sqrt[6]{\frac{3^3\times 2^2}{2^6}}=$$
$$\frac{2}{3}\times\frac{1}{2}\sqrt[6]{27\times 4}=\frac{1}{3}\sqrt[6]{108}.$$

3. Divide $\sqrt{2a^3b}$ by $\sqrt[4]{a^3b^3}$.

$$\frac{\sqrt{2a^3b}}{\sqrt[4]{a^3b^3}}=\frac{\sqrt[4]{4a^6b^2}}{\sqrt[4]{a^3b^3}}=\sqrt[4]{\frac{4a^6b^2}{a^3b^3}}=\sqrt[4]{\frac{4a^3}{b}}=\sqrt[4]{\frac{4a^3}{b}\times\frac{b^3}{b^3}}=\sqrt[4]{\frac{4a^3b^3}{b^4}}$$
$$=\frac{1}{b}\sqrt[4]{4a^3b^3}.$$

EXERCISE 87

Divide:

1. $\sqrt{27}$ by $\sqrt{3}$.
2. $10\sqrt{32}$ by $5\sqrt{2}$.
3. $\sqrt{50}$ by $2\sqrt{5}$.
4. $\sqrt[3]{16}$ by $\sqrt[3]{2}$.
5. $\sqrt[5]{729}$ by $\sqrt[5]{81}$.
6. $\sqrt{\frac{1}{2}}$ by $\sqrt{\frac{2}{3}}$.
7. $\frac{1}{2}\sqrt[3]{\frac{1}{2}}$ by $\frac{1}{3}\sqrt[3]{\frac{1}{4}}$.
8. $3\sqrt{5}$ by $\sqrt{40}$.
9. $\sqrt{\frac{6}{11}}$ by $\sqrt{\frac{22}{27}}$.
10. $\sqrt{abc}$ by $\sqrt{cd}$.
11. $\sqrt[3]{4}$ by $\sqrt{2}$.
12. $\sqrt[3]{10}$ by $\sqrt{5}$.
13. $\sqrt{24}$ by $\sqrt[4]{36}$.
14. $\sqrt{\frac{9}{45}}$ by $\sqrt[4]{8\frac{1}{3}}$.
15. $\sqrt[3]{2a}$ by $\sqrt{a^3}$.
16. $\sqrt{2ax}$ by $\sqrt[3]{a^2x^2}$.
17. $\sqrt[4]{0.081}$ by $\sqrt{10}$.
18. $9\sqrt[3]{3}$ by $3\sqrt{2}$.
19. $\sqrt[4]{9a^2b^2}$ by $\sqrt[3]{3ab}$.
20. $2\sqrt{\frac{2}{3}}$ by $\sqrt[3]{\frac{3}{4}}$.

125. As every fraction is an indicated division, any division may be regarded as a fraction. The division by radicals may therefore be expressed as a fraction and the problem considered a *rationalizing of the denominator.*

1. Divide 5 by $3\sqrt{3}$.

$$5\div 3\sqrt{3}=\frac{5}{3\sqrt{3}}.$$

The factor needed to make the denominator rational is evidently $\sqrt{3}$. Hence multiply numerator and denominator by $\sqrt{3}$.

$$\frac{5}{3\sqrt{3}}\times\frac{\sqrt{3}}{\sqrt{3}}=\frac{5\sqrt{3}}{9}.$$

2. Divide $\sqrt{2} - \sqrt{3}$ by $\sqrt{5} - \sqrt{2}$.

$$\frac{\sqrt{2} - \sqrt{3}}{\sqrt{5} - \sqrt{2}} \times \frac{\sqrt{5} + \sqrt{2}}{\sqrt{5} + \sqrt{2}} = \frac{\sqrt{10} - \sqrt{15} + 2 - \sqrt{6}}{5 - 2} = \frac{2 - \sqrt{6} + \sqrt{10} - \sqrt{15}}{3}.$$

When the divisor is a binomial, the rationalizing factor will always be the divisor with the sign between the terms changed, since the product of the sum and difference of two numbers produces the difference of their squares, a rational quantity.

Hence to rationalize the denominator of a fraction,
Multiply numerator and denominator by the necessary rationalizing factor.

EXERCISE 88

Divide:

1. 2 by $5\sqrt{2}$.
2. a by $b\sqrt{a}$.
3. $\sqrt{8}$ by $3\sqrt{18}$.
4. $\sqrt{x} + \sqrt{y}$ by $\sqrt{xy}$.
5. 5 by $3 - \sqrt{2}$.
6. 1 by $\sqrt{5} - \sqrt{6}$.
7. $2\sqrt{3} - 1$ by $2\sqrt{3} + 1$.
8. $\sqrt{x} + \sqrt{y}$ by $\sqrt{x} - \sqrt{y}$.
9. $\sqrt{7} - \sqrt{11}$ by $\sqrt{11} - \sqrt{3}$.
10. $3\sqrt{2} + 5\sqrt{3}$ by $4\sqrt{2} - 3\sqrt{3}$.

Rationalize the denominators of the following fractions and find the value of the fractions to four decimal places, having given

$$\sqrt{2} = 1.41421, \sqrt{3} = 1.73205 \text{ and } \sqrt{5} = 2.23607.$$

11. $\dfrac{7}{\sqrt{2}}$.
12. $\dfrac{10}{\sqrt{3}}$.
13. $\dfrac{8}{\sqrt{5}}$.
14. $\dfrac{1}{\sqrt{8}}$.
15. $\dfrac{1}{\sqrt{12}}$.
16. $\dfrac{1}{\sqrt{125}}$.
17. $\dfrac{3 - 3\sqrt{2}}{\sqrt{2} + 1}$.
18. $\dfrac{5 - 2\sqrt{3}}{4 - 5\sqrt{3}}$.
19. $\dfrac{11 + 2\sqrt{5}}{3 - \sqrt{5}}$.

INVOLUTION AND EVOLUTION OF RADICALS

126. By using fractional exponents as explained under the Theory of Exponents, any power or root of a radical may easily be found.

1. $(3\sqrt{x})^3 = (3x^{\frac{1}{2}})^3 = 3^3x^{\frac{3}{2}} = 27x\sqrt{x}$.
2. $\sqrt{2x\sqrt[3]{y^2}} = (2x\sqrt[3]{y^2})^{\frac{1}{2}} = (2xy^{\frac{2}{3}})^{\frac{1}{2}} = 2^{\frac{1}{2}}x^{\frac{1}{2}}y^{\frac{1}{3}} = 2^{\frac{3}{6}}x^{\frac{3}{6}}y^{\frac{2}{6}} = \sqrt[6]{8x^3y^2}$.
3. $\sqrt[6]{64\sqrt[5]{32x^{20}}} = (64\sqrt[5]{32x^{20}})^{\frac{1}{6}} = [2^6\cdot(2^5x^{20})^{\frac{1}{5}}]^{\frac{1}{6}} = (2^6\cdot 2^1x^4)^{\frac{1}{6}} = 2\cdot 2^{\frac{1}{6}}x^{\frac{4}{6}} = 2\sqrt[6]{2x^4}$.

EXERCISE 89

Simplify:

1. $(2\sqrt{x})^2$.
2. $(\sqrt[3]{a^2})^4$.
3. $(2\sqrt[4]{x^2})^5$.
4. $(\sqrt[5]{y^3})^{15}$.
5. $\sqrt{\sqrt[3]{a^2b^2}}$.
6. $\sqrt[3]{\sqrt[5]{(x-y)^{15}}}$.
7. $\sqrt[5]{\sqrt[4]{(a+b)^{20}}}$.
8. $\sqrt[4]{\sqrt[3]{(m-n)^{24}}}$.
9. $(\sqrt[5]{a^2-b^2})^{10}$.
10. $\sqrt[3]{\sqrt{64}}$.
11. $\sqrt[4]{16\sqrt[3]{625}}$.
12. $\sqrt[7]{\sqrt[5]{729x^{14}}}$.

QUADRATIC SURDS

127. A **Quadratic Surd** is a surd of the second order, or an indicated square root of an imperfect square. A **Binomial Surd** is a binomial, one or both of whose terms are surds.

The following **Properties of Quadratic Surds** are necessary in the solution of certain problems.

1. *The product or quotient of two dissimilar quadratic surds will be a quadratic surd.*

Thus $\sqrt{xy} \times \sqrt{xyz} = xy\sqrt{z}$.

And $\sqrt{xyz} \div \sqrt{xy} = \sqrt{z}$.

Since the surds are dissimilar, they cannot have all the factors under the radical sign alike. Therefore the product or quotient must contain the first power of at least one factor and hence will be a surd.

2. *The sum or difference of two dissimilar quadratic surds cannot be a rational number, nor can it be expressed as a single surd.*

Thus $\sqrt{x} \pm \sqrt{y}$ cannot equal z.

If $\sqrt{x} \pm \sqrt{y} = z$,

By squaring, $x \pm 2\sqrt{xy} + y = z^2$.

By transposing, $\pm 2\sqrt{xy} = z^2 - x - y$.

But since $2\sqrt{xy}$ is a surd, it cannot equal the rational number $z^2 - x - y$.

Therefore $\sqrt{x} \pm \sqrt{y}$ cannot equal z.

Further, since $\sqrt{x}$ and $\sqrt{y}$ are dissimilar surds, they cannot be combined by addition or subtraction and hence cannot equal a single surd.

3. *A quadratic surd cannot equal the sum of a rational number and a surd.*

That is $\sqrt{x}$ cannot equal $\sqrt{y} + z$.

If $\sqrt{x} = z + \sqrt{y}$.

By squaring, $x = z^2 + 2z\sqrt{y} + y$.

By transposing, $2z\sqrt{y} = x - y - z^2$

which is impossible, as shown in the preceding case, hence $\sqrt{x}$ cannot equal $z + \sqrt{y}$.

4. *If $a + \sqrt{b} = c + \sqrt{d}$, then $a = c$, and $b = d$.*

By transposition, $a - c = \sqrt{d} - \sqrt{b}$.

If a is not equal to c, the difference $a - c$ will be a rational number and cannot equal $\sqrt{d} - \sqrt{b}$.

Therefore a must equal c, and $a - c = 0$.

Hence $\sqrt{d} - \sqrt{b} = 0$.

or $\sqrt{d} = \sqrt{b}$

and $d = b$.

This proves, that in any equation containing both rational numbers and surds, the rational numbers on one side of the equation must equal the rational numbers on the other side, and similarly, the surds on one side must equal the surds on the other side.

5. *If $\sqrt{x + \sqrt{y}} = \sqrt{a} + \sqrt{b}$, then $\sqrt{x - \sqrt{y}} = \sqrt{a} - \sqrt{b}$.*

Squaring the first equation,

$$x + \sqrt{y} = a + 2\sqrt{ab} + b.$$

From the preceding principle,

$$x = a + b \qquad (1)$$

$$\text{and } \sqrt{y} = 2\sqrt{ab} \qquad (2)$$

Subtracting (2) from (1)

$$x - \sqrt{y} = a - 2\sqrt{ab} + b$$

Extracting the square root of both sides,

$$\sqrt{x - \sqrt{y}} = \sqrt{a} - \sqrt{b}.$$

SQUARE ROOT OF A BINOMIAL SURD

128. 1. Extract the square root of $11 + 6\sqrt{2}$.

Since the square of a binomial must have the form $a^2 \pm 2ab + b^2$, the square root of a binomial surd can be found by inspection if it can be written in the form of the square of a binomial.

The middle term must have a coefficient 2, hence $6\sqrt{2}$ may be transformed into $2 \times 3\sqrt{2} = 2\sqrt{9 \times 2} = 2\sqrt{18}$. Since the middle term of the square must be twice the product of the square roots of the first and third terms, 11 must be broken up into two parts whose product will be 18.

Hence $\quad 11 + 6\sqrt{2} = 9 + 2\sqrt{18} + 2.$

Since $\sqrt{a^2 \pm 2ab + b^2} = a \pm b$

$$\sqrt{9 + 2\sqrt{18} + 2} = \sqrt{9} + \sqrt{2} = 3 + \sqrt{2}.$$

2. Extract the square root of $10 + \sqrt{91}$.

As this binomial surd cannot be arranged conveniently in the form of the square of a binomial, a formal method of solution is indicated.

Let $\sqrt{x} + \sqrt{y} = \sqrt{10 + \sqrt{91}}$ (1)

Then by Principle 5,

$$\sqrt{x} - \sqrt{y} = \sqrt{10 - \sqrt{91}} \quad (2)$$

Multiply (1) by (2),

$$x - y = \sqrt{100 - 91} = \sqrt{9} = 3 \quad (3)$$

Squaring (1),

$$x + 2\sqrt{xy} + y = 10 + \sqrt{91} \quad (4)$$

Since the rational quantities on one side equal the rational quantities on the other side of the equation,

$x + y = 10$ (5)

But $\quad x - y = 3$ (3)

By addition, $\quad 2x = 13$

and $\quad x = \frac{13}{2}$

Subtracting (3) from (5),

$2y = 7$

and $\quad y = \frac{7}{2}$

Substituting the values of x and y in (1),

$$\sqrt{10 + \sqrt{91}} = \sqrt{\tfrac{13}{2}} + \sqrt{\tfrac{7}{2}}$$

Rationalizing these fractions,

$$\sqrt{10 + \sqrt{91}} = \sqrt{\tfrac{26}{4}} + \sqrt{\tfrac{14}{4}} = \tfrac{1}{2}\sqrt{26} + \tfrac{1}{2}\sqrt{14}.$$

EXERCISE 90

Find the square root of:

1. $7 + 4\sqrt{3}$.
2. $8 - 6\sqrt{5}$.
3. $19 + 2\sqrt{48}$.
4. $14 - \sqrt{180}$.
5. $5 + 2\sqrt{6}$.
6. $5 - \sqrt{24}$.
7. $61 - 28\sqrt{3}$.
8. $70 + 6\sqrt{125}$.
9. $2\sqrt{35} + 12$.
10. $2 - \sqrt{3}$.
11. $13 + \sqrt{168}$.
12. $50 + 20\sqrt{10}$.
13. $58 - 12\sqrt{20}$.
14. $62 + 10\sqrt{37}$.
15. $2 - \sqrt{7}$.

EQUATIONS CONTAINING RADICALS

129. Equations containing radicals are usually solved by involution. An equation containing a single radical is arranged to have the radical alone on one side, and then the equation is raised to the power corresponding to the order of the radical.

1. Solve $\sqrt{x^2 - 3} + 3 = x$.

Transpose 3, $\sqrt{x^2 - 3} = x - 3$

Squaring, $x^2 - 3 = x^2 - 6x + 9$

$6x = 12$

$\therefore x = 2$

When two or more radicals are involved, two or more applications of involution may be necessary.

2. Solve $\sqrt{4x - 4} + \sqrt{3x} = \sqrt{3x + 7} + \sqrt{4x - 3}$.

Squaring, $4x + 4 + 2\sqrt{3x(4x + 4)} + 3x = 3x + 7 +$
$2\sqrt{(3x + 7)(4x + 3)} + 4x - 3$

Transposing and collecting like terms,

$$2\sqrt{3x(4x - 4)} = 2\sqrt{(3x + 7)(4x - 3)}$$

Dividing by 2, $\sqrt{3x(4x - 4)} = \sqrt{(3x + 7)(4x - 3)}$

Squaring, $3x(4x - 4) = (3x + 7)(4x - 3)$

Simplifying, $12x^2 - 12x = 12x^2 + 19x - 21$

Transposing and collecting like terms,

$$-31x = -21$$

Dividing by -31, $x = \frac{21}{31}$

3. Solve $\sqrt{m+\sqrt{x}}+\sqrt{m-\sqrt{x}}=\sqrt{x}$.

Squaring, $m+\sqrt{x}+2\sqrt{m^2-x}+m-\sqrt{x}=x$

Transposing and collecting,

$$2\sqrt{m^2-x}=x-2m$$

Squaring, $$4(m^2-x)=x^2-4mx+4m^2$$

$$4m^2-4x=x^2-4mx+4m^2$$

Transposing and collecting,

$$x^2+4x-4mx=0$$

Dividing by x,

$$x+4-4m=0$$

$$\therefore x=4m-4$$

EXERCISE 91

Solve:

1. $\sqrt{x+5}=3$.
2. $3\sqrt{2x-5}=\sqrt{27}$.
3. $\sqrt[3]{2x+3}=5$.
4. $2\sqrt{3x-2}=\sqrt{2x-3}$.
5. $3-2\sqrt{y}=4$.
6. $5-\sqrt[3]{2x}=3$.
7. $\sqrt[3]{5x+3}=\sqrt[3]{3x+5}$.
8. $\sqrt{y}=\sqrt{7}+\sqrt{y+7}$.
9. $\sqrt{\frac{5}{4}+x}-\frac{3}{2}=\sqrt{x}$.
10. $\sqrt{x+15}-\sqrt{x-3}+3=0$.
11. $\sqrt{x^2+7}-7=x$.
12. $\sqrt{x+9}-5=5-\sqrt{x-7}$.
13. $\dfrac{\sqrt{y}+2}{\sqrt{y}-3}=\dfrac{\sqrt{y}+3}{\sqrt{y}-1}$.
14. $\sqrt{mx}-3=7+\frac{1}{3}\sqrt{mx}-\frac{1}{2}$.
15. $\dfrac{\sqrt{1+x}+1}{\sqrt{1+x}-1}=5$.
16. $\sqrt{5+3x}-\sqrt{3x}=\dfrac{7}{\sqrt{5+3x}}$.
17. $\sqrt{5x+2}+\sqrt{2x+5}=\sqrt{2x-7}+\sqrt{5x}$.
18. $\sqrt{12+\sqrt{3+\sqrt{x-5}}}=4$.

REVIEW EXERCISE XV

Simplify:

1. $3\sqrt{\frac{2}{3}} - 2\sqrt{\frac{1}{6}} + \sqrt[4]{36} - 5\sqrt{180}$.
2. $\sqrt{\frac{1}{2}} - \sqrt{\frac{2}{9}} + \frac{1}{4}\sqrt{\frac{9}{2}} - \frac{1}{2}\sqrt{\frac{25}{32}}$.
3. $\sqrt[3]{54} + 6\sqrt[3]{\frac{1}{4}} - 2\sqrt[6]{4} + (3x)^0$.
4. $30\sqrt{\frac{1}{2}} - \frac{9}{2}\sqrt{8} + 9\sqrt{84\frac{1}{2}}$.
5. $\dfrac{a-1}{2}\sqrt[3]{\dfrac{8(a+1)}{(a-1)^2}}$.
6. $\dfrac{a}{a^2-b^2}\sqrt{\dfrac{a^3c - 2a^2bc + ab^2c}{b^3}}$.
7. $2\sqrt{\frac{2}{25}} - \sqrt[6]{125} + \sqrt[8]{\frac{16}{81}} - \sqrt{\frac{2}{3}} + \frac{1}{5}\sqrt{90} - \sqrt[4]{36} + (\sqrt[4]{5})^2$.
8. $\sqrt[10]{\frac{100}{63}} \div \sqrt[5]{\frac{10}{21}}$.
9. $\dfrac{5a}{6}\sqrt[3]{\dfrac{x}{5}} + \dfrac{a}{15b}\sqrt[3]{200b^3x} - 4\sqrt[3]{\dfrac{25a^3x}{512}}$.
10. $\sqrt{2ax} \times \sqrt[5]{2a^4x^2}$.
11. $\sqrt[3]{3} \times 3\sqrt{3}$ and $\sqrt[4]{4x^2y^2} \div \sqrt[3]{2xy}$.
12. Subtract $\dfrac{1}{2+\sqrt{3}}$ from $\dfrac{3}{5-2\sqrt{3}}$ and express result with rational denominator.

Rationalize the denominators:

13. $\dfrac{4\sqrt{2}+6\sqrt{3}}{3\sqrt{3}-2\sqrt{2}}$.
14. $\dfrac{\sqrt{8}+\sqrt{7}}{\sqrt{7}-\sqrt{2}}$.
15. $\dfrac{a^2 - a\sqrt{b}+b}{a-\sqrt{b}}$.
16. $\dfrac{\sqrt{8}-\sqrt{6}}{\sqrt{8}+\sqrt{6}}$.
17. $\dfrac{x\sqrt{y}+y\sqrt{x}}{y\sqrt{x}-x\sqrt{y}}$.
18. $\dfrac{\sqrt{x+1}-2}{\sqrt{x+1}+2}$.
19. $\dfrac{\sqrt{x-3}+\sqrt{x}}{\sqrt{x-3}-\sqrt{x}}$.

Extract the square root of:

20. $30 - 20\sqrt{2}$.
21. $15 - 4\sqrt{14}$.
22. $11 - 6\sqrt{2}$.
23. $117 - 36\sqrt{10}$.
24. $11 + 4\sqrt{7}$.
25. $5 + 2\sqrt{6}$.

Solve the equations:

26. $\sqrt{4x+9} - \sqrt{x-1} = \sqrt{x+6}$.

27. $\sqrt{x+20} - \sqrt{x-1} = 3$.

28. $\sqrt{x+5} + \sqrt{3x+4} = \sqrt{12x+1}$.

29. $\sqrt{15+x} = 3\sqrt{5} - \sqrt{x}$.

30. $\sqrt{3} + \sqrt{9x+1} = 3\sqrt{x}$.

31. $\sqrt{12x-11} - \sqrt{3x+1} = \sqrt{3x-6}$.

32. $2\sqrt{x} - \sqrt{4x-22} - \sqrt{2} = 0$.

33. $\sqrt{x-5} - \sqrt{4x-2} + \sqrt{x+3} = 0$.

34. $\sqrt{9x-5} - 2\sqrt{4x-15} - \sqrt{x-5} = 0$.

35. $\dfrac{3}{\sqrt{x-3}} - \sqrt{x-3} = \sqrt{x}$.

CHAPTER XVI

IMAGINARY NUMBERS

130. As any even power of a number is positive, any expression involving an indicated even root of a negative number is called an **imaginary expression.**

To distinguish them from imaginary numbers, all other numbers are called **real numbers.**

The algebraic sum of a *real number* and an *imaginary number* is called a **complex number.**

In this chapter, only indicated *square roots* of negative numbers are considered, as any indicated even root of a negative number may be made to assume a form involving only a square root of a negative number.

Since $\sqrt{ab} = \sqrt{a} \times \sqrt{b}$, when a and b are positive, it is assumed that this is also true when either a or b is negative.

That is, $\sqrt{-9} = \sqrt{9} \times \sqrt{-1} = 3\sqrt{-1}$.

So that any imaginary square root may be factored into a real number and the imaginary factor $\sqrt{-1}$.

$\sqrt{-1}$ is called the **imaginary unit** and for brevity is sometimes expressed by i.

Since the square of the square root of a number is the number itself, $(\sqrt{-1})^2 = -1$. This is the important fact underlying operations with imaginary numbers.

The successive powers of $\sqrt{-1}$ are,

$$\sqrt{-1} = +\sqrt{-1}$$
$$(\sqrt{-1})^2 = -1$$
$$(\sqrt{-1})^3 = (\sqrt{-1})^2(\sqrt{-1}) = (-1)(\sqrt{-1}) = -\sqrt{-1}$$
$$(\sqrt{-1})^4 = (\sqrt{-1})^2(\sqrt{-1})^2 = (-1)(-1) = +1$$
$$(\sqrt{-1})^5 = (\sqrt{-1})^4(\sqrt{-1}) = (+1)(\sqrt{-1}) = +\sqrt{-1}$$

and so on.

Remembering that $(\sqrt{-1})^2 = -1$ and $(\sqrt{-1})^4 = +1$, the value of any power of $\sqrt{-1}$ can easily be found.

CARL FRIEDRICH GAUSS

(After a Lithograph From a Painting by Jensen)

Born at Braunschweig, 1777. Died at Goettingen, 1855.

Gauss was a physicist and astronomer and the greatest mathematician of modern times. He is responsible for the acceptance and understanding of imaginary quantities and complex numbers.

Thus, $(\sqrt{-1})^{17} = (\sqrt{-1})^{16}(\sqrt{-1})^{1} = (+1)(\sqrt{-1}) = +\sqrt{-1}$.

Since every fourth power of $\sqrt{-1}$ yields a factor $+1$.

Similarly $(\sqrt{-1})^{15} = (\sqrt{-1})^{12}(\sqrt{1})^{3} = (+1)(\sqrt{-1})^{2}(\sqrt{-1}) = (+1)(-1)(\sqrt{-1}) = -\sqrt{-1}$.

Hence any exponent of $\sqrt{-1}$ may be reduced by any multiple of 4. As, $(\sqrt{-1})^{17} = (\sqrt{-1})^{1}$ and $(\sqrt{-1})^{15} = (\sqrt{-1})^{3}$.

OPERATIONS WITH IMAGINARIES

131. 1. Add $3 + 6\sqrt{-1}$ and $5 - \sqrt{-9}$.

$$3 + 6\sqrt{-1} + 5 - \sqrt{-9} = 3 + 6\sqrt{-1} + 5 - \sqrt{+9}\sqrt{-1} = 3 + 6\sqrt{-1} + 5 - 3\sqrt{-1} = 8 + 3\sqrt{-1}$$

2. Multiply $\sqrt{-3} + 2\sqrt{-2}$ by $3\sqrt{-3} - 5\sqrt{-2}$.

Factoring out the imaginary unit,

$$\begin{aligned}
&(\sqrt{3}\sqrt{-1} + 2\sqrt{2}\sqrt{-1})(3\sqrt{3}\sqrt{-1} - 5\sqrt{2}\sqrt{-1}) = \\
&[(\sqrt{3} + 2\sqrt{2})\sqrt{-1}][(3\sqrt{3} - 5\sqrt{2})\sqrt{-1}] = \\
&-1(\sqrt{3} + 2\sqrt{2})(3\sqrt{3} - 5\sqrt{2}) = \\
&-1(9 + 6\sqrt{6} - 5\sqrt{6} - 20) = -1(\sqrt{6} - 11) = 11 - \sqrt{6}
\end{aligned}$$

3. Multiply $\sqrt{6} + \sqrt{-3}$ by $\sqrt{6} - \sqrt{-3}$.

Factoring out the imaginary unit,

$$\begin{array}{llll}
 & \sqrt{6} + \sqrt{3}\sqrt{-1} & & \\
 & \sqrt{6} - \sqrt{3}\sqrt{-1} & & \\
\hline
6 & + \sqrt{18}\sqrt{-1} & & \\
 & - \sqrt{18}\sqrt{-1} & - 3(-1) & \\
\hline
6 & & + 3 & = 9
\end{array}$$

4. Divide $\sqrt{6} - \sqrt{-8}$ by $\sqrt{5} + \sqrt{-6}$.

$$\begin{aligned}
&\frac{\sqrt{6} - \sqrt{-8}}{\sqrt{5} + \sqrt{-6}} = \frac{\sqrt{6} - \sqrt{8}\sqrt{-1}}{\sqrt{5} + \sqrt{6}\sqrt{-1}} = \\
&\frac{(\sqrt{6} - \sqrt{8}\sqrt{-1})(\sqrt{5} - \sqrt{6}\sqrt{-1})}{(\sqrt{5} + \sqrt{6}\sqrt{-1})(\sqrt{5} - \sqrt{6}\sqrt{-1})} = \\
&\frac{\sqrt{30} - \sqrt{40}\sqrt{-1} - \sqrt{36}\sqrt{-1} + \sqrt{48}(-1)}{\sqrt{25} - \sqrt{36}(-1)} = \\
&\frac{\sqrt{30} - 2\sqrt{-10} - 6\sqrt{-1} - 4\sqrt{3}}{5 + 6} = \\
&\frac{\sqrt{30} - 2\sqrt{-10} - 6\sqrt{-1} - 4\sqrt{3}}{11}.
\end{aligned}$$

EXERCISE 92

Simplify:

1. $(\sqrt{-1})^7$. 2. $(\sqrt{-1})^{13}$. 3. $(\sqrt{-1})^{12}$. 4. $(\sqrt{-1})^{22}$
5. $(\sqrt{-1})^{19}$.

Add:

6. $\sqrt{-36} + \sqrt{-81} - \sqrt{144}$.
7. $4\sqrt{-27} + \sqrt{-12} - 2\sqrt{-3}$.
8. $\sqrt{-2} + 3\sqrt{-3} - \sqrt{-32} - \sqrt{-12}$.
9. $\sqrt{-x^2} + \sqrt{+16x^2} - 3\sqrt{-x^2}$.
10. $\sqrt{-a^4} - \sqrt{-4a^2b^2} + \sqrt{-b^4}$.

Multiply:

11. $\sqrt{-5}$ by $\sqrt{-7}$. 12. $\sqrt{-25}$ by $\sqrt{-9}$.
13. $\sqrt{-16}$ by $-\sqrt{-8}$.
14. $3 + \sqrt{-2}$ by $2 - \sqrt{-3}$.
15. $a\sqrt{-x} + b\sqrt{-y}$ by $a\sqrt{-x}.- b\sqrt{-y}$.

Divide:

16. $\sqrt{-15}$ by $\sqrt{-5}$.
17. x by $\sqrt{-1}$.
18. $-\sqrt{-x^2}$ by $\sqrt{-y^2}$.
19. $\sqrt{-5} + \sqrt{-6}$ by $\sqrt{-6} - \sqrt{-8}$.
20. $8 + 3\sqrt{-7}$ by $3 - 8\sqrt{-7}$.

REVIEW EXERCISE XVI

1. Simplify:

$$\sqrt{-20} - 9\sqrt{27} + 5\sqrt{\frac{-1}{25}} + 3\sqrt{-3} + 2\sqrt[6]{9}.$$

2. Simplify, $5\sqrt{-\frac{1}{2}} - 3\sqrt{-\frac{9}{2}} - 4\sqrt{-50} - \sqrt{-200}$.

3. Rationalize the denominator of $\dfrac{1}{3 - \sqrt{-2}}$.

4. Multiply $3\sqrt{-5} - 4\sqrt{-6}$ by $6\sqrt{-5} + 8\sqrt{-6}$.

5. Rationalize the denominator of $\dfrac{\sqrt{-2} - 3\sqrt{-3}}{\sqrt{-2} + 2\sqrt{-3}}$.

6. Simplify, $\dfrac{m + n\sqrt{-1}}{m - n\sqrt{-1}} + \dfrac{m - n\sqrt{-1}}{m + n\sqrt{-1}}$.

7. Rationalize the denominator of $\frac{7}{2-\sqrt{-3}}$.

8. Simplify, $\frac{2}{\sqrt{2}-1}-(1-\sqrt{-2})^2$.

9. Simplify,

$$4^{-1}-4^{-\frac{1}{2}}+4^0-\frac{1}{\sqrt{8}}+\sqrt{50}-\sqrt{-4}\sqrt{-9}.$$

10. Simplify, $\frac{(x-\sqrt{-1})^2}{x+\sqrt{-1}}+\frac{(x+\sqrt{-1})^2}{x-\sqrt{-1}}$.

11. Find the value of $\left(\frac{-1+\sqrt{-3}}{2}\right)^3+\left(\frac{-1-\sqrt{-3}}{2}\right)^3$.

12. Find the value of

$$(1+\sqrt{-3})(1+\sqrt{-3})(1+\sqrt{-3}).$$

13. Find the value of $\frac{7+3\sqrt{-5}}{7-3\sqrt{-5}}+\frac{7-3\sqrt{-5}}{7+3\sqrt{-5}}$.

CHAPTER XVII

QUADRATIC EQUATIONS

132. An equation, that contains the square of the unknown number, but no higher power, is called an equation of the **second degree** or a **quadratic equation.**

An equation containing only the square of the unknown number is called a **pure quadratic** or **incomplete quadratic,** as the first power of the unknown is lacking.

If the equation contains the first power of the unknown as well as the square, it is called an **affected quadratic** or a **complete quadratic.**

Thus $x^2 = a$, is a pure quadratic, while $2x^2 + 3x = 7$ is an affected quadratic.

Every simple equation or equation of the first degree has one root. As, $x - 3 = 0$ has $x = 3$ as its root, and $x + 4 = 0$ has $x = -4$ as its root. Hence, the product of these two simple equations, $(x - 3)(x + 4) = 0$ or the quadratic equation $x^2 + x - 12 = 0$, has two roots, $+3$ and -4.

In general, *the number of roots of any equation is numerically equal to the degree of the equation.*

PURE QUADRATIC EQUATIONS

133. Any pure quadratic equation can be made to assume the form $x^2 = a$.

Taking the square root of each side of the equation, $\pm x = \pm \sqrt{a}$.

Separating the double signs, we get four equations,

$$+x = +\sqrt{a} \quad (1) \qquad -x = +\sqrt{a} \quad (2).$$
$$+x = -\sqrt{a} \quad (3) \qquad -x = -\sqrt{a} \quad (4).$$

From (1) and (4) we have $x = +\sqrt{a}$.

From (2) and (3) we have $x = -\sqrt{a}$.

Hence, in solving a quadratic, it is unnecessary to place the ± sign before the left side of the equation, as no roots will be lost by this omission.

1. Solve $2x^2 - 3 = x^2 + 6$.

Transposing and combining like terms,

$$x^2 = 9.$$

Extracting the square root,

$$x = \pm 3.$$

2. Solve $2x^2 + 3 = x^2 + 6$.

Transposing and collecting,

$$x^2 = 3.$$

Extracting the square root,

$$x = \pm\sqrt{3}.$$

3. Solve $2x^2 + 3 = 3x^2 + 6$.

Solving as before,

$$x^2 = -3$$

$$\therefore x = \pm\sqrt{-3}.$$

In the first problem, the two roots were found exactly and are called **rational roots.**

In the second problem, the two roots, $\pm\sqrt{3}$, can only be found approximately and are called **surd roots.**

Both rational roots and surd roots are called **real roots.**

In the third problem, the roots, $\pm\sqrt{-3}$, being imaginary numbers, are called **imaginary roots.**

EXERCISE 93

1. $2x^2 - 4 = 44 - x^2$.
2. $3x^2 - 3 = x^2 + 5$.
3. $5x^2 - 3 = 6x^2 - 12$.
4. $\frac{2}{5}x^2 + 2 = 12$.
5. $\dfrac{x^2+3}{4} = 7$.
6. $\dfrac{x^2-2}{3} = \dfrac{x^2+4}{4}$.
7. $\dfrac{2x^2+3}{5} + \dfrac{3x^2-2}{7} = 4$.
8. $\dfrac{3x^2+5}{3} - \dfrac{4-x^2}{4} = 7$.
9. $\dfrac{2}{3x^2} - \dfrac{4}{5x^2} = 4$.
10. $4x^2 + 7x = 5x - 3x^2 + 28 + 2x$.
11. $2(x+3)(x-2) = (x+5)(x-3) + 25$.

12. $\frac{1}{x+2} - \frac{1}{x-2} = \frac{1}{5}$

13. $\frac{x+2}{x-2} + \frac{x-2}{x+2} = 2.$

14. $\frac{x+3}{x-1} + \frac{x-3}{x+1} = -1.$

15. $\frac{2x^2+5}{7} - \frac{x^2+9}{3} + \frac{3x^2-4}{2} = 11.$

16. $\frac{8x^2-3}{16} - \frac{4x^2+5}{8} = \frac{3-4x^2}{7x^2-5}.$

17. $2x^2 - a^2 = 11a^2 - x^2.$

18. $3a^2x^2 - 3b^2 = a^2x^2 + 5b^2.$

19. $ax^2 - b = c.$

20. $ax^2 + c = cx^2 + a.$

21. $\frac{x^2+b^2}{b} = 5b.$

22. $x^2 + 2ax + b = a(1+2x).$

23. $\frac{1}{x^2} - \frac{1}{a^2} = \frac{17a^2 - 3x^2}{a^2x^2}.$

24. $(x+a)(x-b) + (x+b)(x-a) = a^2 + b^2.$

25. $\frac{x+a}{x-a} + \frac{x-a}{x+a} = 2.$

Solve for each letter involved:

26. $s = \frac{1}{2}gt^2.$

27. $R = K\frac{l}{d^2}.$

28. $E = \frac{1}{2}mv^2.$

29. $t = 3.142\sqrt{\frac{l}{g}}.$

30. $f = K\frac{m_1m_2}{r^2}.$

31. $\frac{I_1}{I_2} = \frac{{d_1}^2}{{d_2}^2}.$

AFFECTED QUADRATIC EQUATIONS

SOLUTION BY FACTORING

134. Every affected quadratic equation may be changed to the form $ax^2 + bx + c = 0$.

Factoring the left member of this equation, furnishes the simplest method of solution.

Thus, $x^2 - 5x - 6 = 0$

Factored, $(x-6)(x+1) = 0.$

But, a product can only be equal to zero, when one of its factors is equal to zero. Hence either

$x - 6 = 0$ or $x + 1 = 0$

If $x - 6 = 0,\qquad x = 6$

If $x + 1 = 0,\qquad x = -1.$

The roots of the equation are therefore, $+6$ and -1.

EXERCISE 94

Solve, by factoring:

1. $x^2 + x - 2 = 0.$	6. $2x^2 + 7x - 15 = 0.$
2. $x^2 - 5x + 6 = 0.$	7. $6x^2 - x = 12.$
3. $x^2 + 9x + 20 = 0.$	8. $4x^2 + 35x = 9.$
4. $x^2 + 5x = 24.$	9. $10v^2 + 10 = 29v.$
5. $2x^2 + 4x - 70 = x^2 + 7.$	10. $3(z^2 - 4) = 16z.$

135. When a quadratic equation cannot be factored readily, one of the following methods is indicated. To obtain the first power of x from x^2, it is necessary to extract the square root, but as the equation also contains the first power of x, the left member of the equation must be changed into a perfect square. This is called **completing the square.**

WHEN THE COEFFICIENT OF X^2 IS UNITY

Any affected quadratic may be made to assume the form $x^2 + bx = c$, by dividing the equation by the coefficient of x^2.

Since $(a + b)^2 = a^2 + 2ab + b^2$, it is evident that the third term of the square, b^2, is the square of half the coefficient of a.

Hence, in the equation, $x^2 + bx = c$, the left member may be made a perfect square, by adding the square of half the coefficient of x.

1. Solve, $x^2 + 4x = 12$.

Adding to both sides of the equation, the square of half the coefficient of x,

$$x^2 + 4x + 4 = 12 + 4 = 16$$

Extracting the square root of each side of the equation,

$$x + 2 = \pm 4$$

Transposing, $x = \pm 4 - 2$
Using the plus sign, $x = +4 - 2 = 2$
Using the minus sign, $x = -4 - 2 = -6$

Hence, the rule,

Simplify the equation.
Divide by the coefficient of x^2.
Add to both sides, the square of half the coefficient of x.
Extract the square root of each side of the equation.
Finally solve the two resulting simple equations.

EXERCISE 95

Solve:

1. $x^2 + 2x = 15$.
2. $x^2 - 8x = 20$.
3. $x^2 - 6x = 16$.
4. $x^2 + 5x = 6$.
5. $3x^2 - 16x = 12$.
6. $10x^2 - 29x = 10$.
7. $4y^2 + 35y = 9$.
8. $6z^2 - z = 12$.
9. $6v^2 + 19v = -15$.
10. $21s^2 + 17s = -2$.

WHEN THE COEFFICIENT OF X^2 IS A SQUARE

136. Since $(2x \pm 3)^2 = 4x^2 \pm 12x + 9$, whenever the first two terms of such a square are given, the third term may be found by *squaring the quotient obtained by dividing the second term by twice the square root of the first term.*

This furnishes the rule for completing the square, when the coefficient of x^2 is a square.

1. Solve $64x^2 + 80x = 75$.

Dividing $80x$ by $16x$, twice the square root of $64x^2$, we get the quotient 5. Hence, complete the square by adding 25 to each side of the equation.

$$64x^2 + 80x + 25 = 100$$

Extracting the square root of each member of the equation,

$$8x + 5 = \pm 10$$

Transposing, $8x = \pm 10 - 5$

$$\therefore 8x = 5 \text{ or } -15$$

Dividing by 8, $x = \frac{5}{8} \text{ or } -\frac{15}{8}$

If the coefficient of x^2 is not a square, multiply the equation by a number that will make the coefficient a square.

2. Solve, $-6x^2 + x = -12$.

Multiply the equation by -6,

$$36x^2 - 6x = 72$$

Complete the square by adding $\left(\frac{6x}{12x}\right)^2 = (\frac{1}{2})^2 = \frac{1}{4}$

$$36x^2 - 6x + \tfrac{1}{4} = 72 + \tfrac{1}{4} = \tfrac{289}{4}$$

Extracting the square root,

$$6x - \tfrac{1}{2} = \pm\tfrac{17}{2}$$

Transpose, $6x = \pm\frac{17}{2} + \frac{1}{2}$

$\therefore 6x = 9$ or -8

Divide by 6, $x = \frac{3}{2}$ or $-\frac{4}{3}$

Note.—If the coefficient of x^2 is negative, multiply the equation by a negative number, since the first term of the square of a binomial must be positive.

EXERCISE 96

Solve:

1. $2x^2 + 5x = -3$.
2. $4x^2 + 11x = 3$.
3. $9x^2 + 6x = 8$.
4. $3x^2 - x = 2$.
5. $5x^2 - 8x = -3$.
6. $-6x^2 + x = -2$.
7. $18x^2 + 9x = 2$.
8. $8z^2 - 37z = 15$.
9. $5y^2 + 13y = 6$.
10. $36x^2 - 19x = 6$.
11. $\frac{x^2}{2} + \frac{35x}{8} = \frac{9}{8}$.
12. $2x^2 - \frac{29}{5}x = 2$.
13. $3z - 16 = \frac{12}{z}$.
14. $v + \frac{19}{6} = -\frac{5}{2v}$.
15. $(3y + 2)(y - 1) - 57 + (y + 2)(2y + 1) = 0$.
16. $2z(z + 5) - (z + 2)(z - 3) = 1$.
17. $\frac{(3x - 5)(2x + 3)}{5} + \frac{x(x - 4)}{3} = 5$.
18. $\frac{2}{x + 1} - \frac{3}{x + 2} = \frac{1}{5}$.
19. $\frac{x + 2}{x - 2} + \frac{x + 3}{x - 3} = -7$.
20. $\frac{3}{5}(4x^2 - 7x - 6) - \frac{1}{10}(2x^2 + 6) = 3x - 4$.

HINDOO METHOD

137. Fractions will be avoided in the solution of an affected quadratic, if the equation is multiplied by *four times the coefficient of* x^2.

1. Solve $2x^2 - 3x = 2$.

Multiply by 8,

$$16x^2 - 24x = 16.$$

According to the preceding method, complete the square by adding $\left(\frac{24x}{8x}\right)^2 = 3^2$, which is the square of the coefficient of x in the original equation.

Hence, $16x^2 - 24x + 9 = 25$

Extract square root, $4x - 3 = \pm 5$

Transpose, $4x = \pm 5 + 3$

$\therefore 4x = 8 \text{ or } -2$

And $x = 2 \text{ or } -\frac{1}{2}$

Hence the following rule,

Simplify the equation.

Multiply the equation by four times the coefficient of x^2.

Complete the square by adding to both sides, the square of the coefficient of x *in the original equation.*

If the coefficient of x is an even number, this method, called the Hindoo method, may be modified so as to reduce the size of the numbers involved.

In this case, *multiply the equation by the coefficient of* x^2 *and complete the square by adding to both sides, the square of half the coefficient of* x *in the original equation.*

2. Solve, $5x^2 - 8x = -3$.

Multiply by 5, $25x^2 - 40x = -15$

Add the square of half the coefficient of x,

$$25x^2 - 40x + 16 = 1$$

Extract square root, $5x - 4 = \pm 1$

Transpose, $5x = \pm 1 + 4$

$\therefore 5x = 5 \text{ or } 3$

And $x = 1 \text{ or } \frac{3}{5}$

EXERCISE 97

Solve:

1. $5x^2 + 9x = 2.$
2. $2x^2 - 5x = 12.$
3. $3x^2 - 17x = -10.$
4. $4x^2 - 5x = 6.$
5. $2x^2 + x = 15.$
6. $9x^2 - 42x = -40.$
7. $16x^2 + 20x = 66.$
8. $16x^2 + 50x = 21.$
9. $6x^2 - x = 6.$
10. $3x^2 - 7x = 6.$
11. $x^2 + \frac{7}{6}x = -\frac{1}{3}.$
12. $\frac{1}{2}x^2 + x + \frac{3}{8} = 0.$
13. $10x - 17 + \dfrac{3}{x} = 0.$
14. $(3x - 1)^2 - (4x - 1)(2x + 1) = 13.$
15. $(x^4 + 1) - (x^2 - 3)(x^2 + 2) - (x - 5) = 0.$
16. $\dfrac{x + 2}{x - 2} + 1 = \dfrac{x - 2}{x + 2}.$
17. $\dfrac{2}{x^2 - 16} - \dfrac{4}{4 - x} = 1 + \dfrac{5}{x + 4}.$
18. $\dfrac{2}{2 - x} + \dfrac{x^2 + x - 1}{8 - x^3} = \dfrac{3x + 4}{x^2 + 2x + 4}$

LITERAL EQUATIONS

138. Literal quadratic equations are solved by the same methods as numerical equations.

Note that every equation must be put into the form of $ax^2 + bx = c$ before any method of completing the square can be applied.

1. Solve $x^2 + (a - b)x = ab.$

Since the coefficient of x^2 is unity, use the first method of completing the square.

Add to both members the square of half the coefficient of x,

$$x^2 + (a - b)x + \left(\frac{a - b}{2}\right)^2 = ab + \frac{a^2 - 2ab + b^2}{4} = \frac{a^2 + 2ab + b^2}{4}$$

Extract square root, $x + \dfrac{a - b}{2} = \pm\dfrac{a + b}{2}$

Transpose, $$x = \pm\frac{a + b}{2} - \frac{a - b}{2}$$

$$\therefore x = \frac{2b}{2} = b \text{ or } -\frac{2a}{2} = -a.$$

2. Solve $2x^2 - 3mx - 5nx + m^2 + 4mn + 3n^2 = 0$.

Transposing, $2x^2 - 3mx - 5nx = -m^2 - 4mn - 3n^2$

Collecting the coefficients of x,

$$2x^2 - (3m + 5n)x = -m^2 - 4mn - 3n^2$$

Use the Hindoo method, by multiplying by 8,

$$16x^2 - 8(3m + 5n)x = -8m^2 - 32mn - 24n^2$$

Complete the square by adding to both sides the square of the coefficient of x

$$16x^2 - 8(3m + 5n)x + (3m + 5n)^2 = -8m^2 - 32mn - 24n^2 + 9m^2 + 30mn + 25n^2$$
$$= m^2 - 2mn + n^2$$

Extract square root, $4x - (3m + 5n) = \pm (m - n)$

Transpose, $4x = \pm (m - n) + (3m + 5n)$

$$\therefore 4x = 4m + 4n \text{ or } 2m + 6n$$

And $x = m + n \text{ or } \frac{m + 3n}{2}$

EXERCISE 98

Solve, by any method:

1. $6x^2 - ax = 35a^2$.
3. $x^2 + 2ax + 3bx = -6ab$.
2. $x^2 - 2mx + m^2 - n^2 = 0$.
4. $8x^2 + 21m^2 = 22mx$.
5. $x^2 - \frac{ax}{2} = \frac{bx}{3} - \frac{ab}{6}$.
6. $\frac{4x^2}{a - b} - 3x = a + b$.
7. $x^2 - 2bx - 3cx + 6c^2 = 0$.
8. $ax^2 + bx = c$.
9. $\frac{x^2}{m - n} - \frac{2m - 3n}{6(m - n)}x = \frac{x}{2} - \frac{2m - 3n}{12(m - n)}$.
10. $\frac{x^2}{ab} + \frac{x}{b^2} = \frac{x}{a^2} + \frac{1}{ab}$.
11. $\frac{mn}{6x} + mnx + \frac{2m^2 + 3n^2}{6} = 0$.
12. $\frac{x}{1 + x} - \frac{x}{x - 1} - m = 0$.
13. $x^2 - \frac{4(mn - 1)}{mn} = \frac{2x}{mn}$.

14. $\dfrac{2bx - b}{2x - 1} + a = x.$

15. $\dfrac{1}{a} + \dfrac{1}{b} + \dfrac{1}{x} = \dfrac{1}{a + b + x}.$

16. $\dfrac{x - c}{x + c} = \dfrac{x - a}{2x + a}.$

17. $\dfrac{x^2 - (a - b)^2}{a} - (x + a - b) = x - 2a + b.$

18. $\dfrac{(a + 2x)(2a + x)}{\dfrac{2a + x}{2a - x} + \dfrac{a - 2x}{a + 2x}} = \dfrac{3}{8}.$

19. $x^2 - 2ax = (b - c + a)(b - c - a).$

20. $x^2 + \dfrac{n - p}{m}x - \dfrac{m - p}{n}x = \dfrac{(m - p)(n - p)}{mn}.$

SOLUTION BY FORMULA

139. Since every affected quadratic equation can be put into the form $ax^2 + bx + c = 0$, the solution of this equation will furnish a *formula* for the solution of any affected quadratic.

Solve $ax^2 + bx + c = 0$.
Transpose, $ax^2 + bx = -c$.
Multiply by $4a$, $4a^2x^2 + 4abx = -4ac$.
Complete the square, $4a^2x^2 + 4abx + b^2 = b^2 - 4ac$.
Extract square root, $2ax + b = \pm\sqrt{b^2 - 4ac}$.
Transpose, $2ax = -b \pm \sqrt{b^2 - 4ac}$.
Divide by $2a$, $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$

In this formula, a represents the coefficient of x^2, b, the coefficient of x and c, the known numbers.

Solve $6x^2 + x = 15$.
Put equation into form $ax^2 + bx + c = 0$

$$6x^2 + x - 15 = 0$$

where, $a = 6$, $b = 1$ and $c = -15$

Substitute in the formula, $x = \dfrac{-1 \pm \sqrt{1 + 360}}{12}.$

$$\text{hence } x = \frac{-1+\sqrt{1+360}}{12} \text{ or } \frac{-1-\sqrt{1+360}}{12}$$

$$= \frac{-1+\sqrt{361}}{12} \text{ or } \frac{-1-\sqrt{361}}{12}$$

$$= \frac{-1+19}{12} \text{ or } \frac{-1-19}{12}$$

$$= \frac{18}{12} \text{ or } -\frac{20}{12}$$

$$= \frac{3}{2} \text{ or } -\frac{5}{3}$$

EXERCISE 99

Solve by formula:

1. $2x^2 - 5x + 2 = 0.$
2. $x^2 + 7x + 12 = 0.$
3. $15x^2 - 2x = 1.$
4. $7x^2 - 10x = 33.$
5. $2x^2 - \frac{1}{2}x - \frac{3}{4} = 0.$
6. $6x^2 + 10x + 6 = 0.$
7. $2x^2 - 5x = 25.$
8. $30x^2 - 61x + 30 = 0.$
9. $abx^2 - acx + b^2x = bc.$
10. $x^2 + px = q.$

GRAPHIC SOLUTION

140. A quadratic equation having real roots may be solved graphically by the principles of graphs previously explained.

1. Let it be required to solve $x^2 + x - 6 = 0$.

If we let $x^2 + x - 6 = y$, the graph of this equation may be found.

Substituting values for x, we find the corresponding values of y.

x	y
0	− 6
1	− 4
2	0
3	6
4	14
− 1	− 6
− 2	− 4
− 3	0
− 4	6
− 5	14

Locate the various points as shown in Fig. I, and connect these points by a smooth curve, which represents the graph of $x^2 + x - 6 = y$.

The roots of the equation, $x^2 + x - 6 = 0$, are those values of x that will make $x^2 + x - 6 = 0$, but since $y = x^2 + x - 6$, any value of x that makes y equal to zero, must be a root of the equation. The ordinate y becomes zero for any point on the X axis, hence the roots are located by the intersections of the graph with the X axis. In Fig. I, the graph cuts the X axis, when $x = 2$ and again when $x = -3$, therefore the roots are 2 and -3.

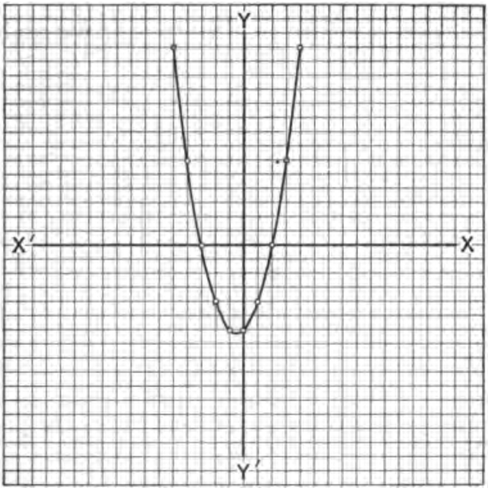


FIG. I.

Since the value of an algebraic expression, like $x^2 + x - 6$, depends upon the value of the letter involved, it is called a **function** of that letter.

Thus, $x^2 + x - 6$ is a *function of x*, expressed by the notation, $f(x)$.

Hence, since $x^2 + x - 6 = f(x)$,

Any value of x that makes $f(x)$ equal to zero will be a root of the equation $x^2 + x - 6 = 0$.

In the graphical solution of this equation, the graph shows the corresponding values of x and $f(x)$, that is, the abscissa of any point is the value of x, and the corresponding ordinate is the value of $f(x)$.

2. Solve $x^2 - 4x + 4 = 0$.

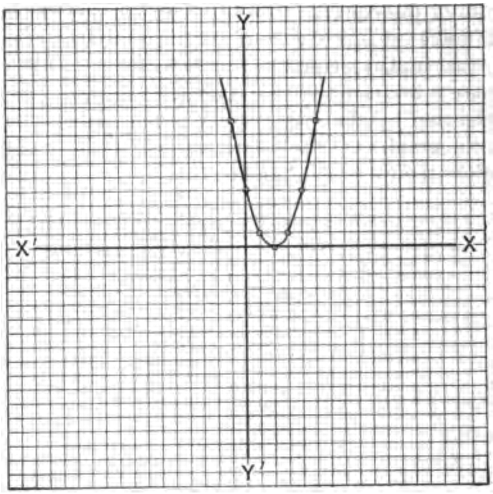


Fig. II.

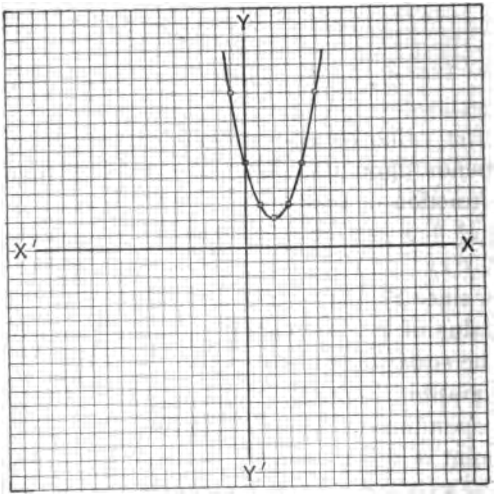


Fig. III.

Plotting the graph of $x^2 - 4x + 4 = y$, Fig. II, we note that the curve touches the X axis at $+2$, but does not cross the axis. This indicates that the two roots of the equation are each equal to $+2$.

3. Solve $x^2 - 4x + 6 = 0$.

Plotting the graph of $x^2 - 4x + 6 = y$, Fig. III, we see that the graph approaches the X axis but does not cut it. This shows that the equation $x^2 + 4x + 6 = 0$ has no real roots, both roots being imaginary, as may be shown by an algebraic solution.

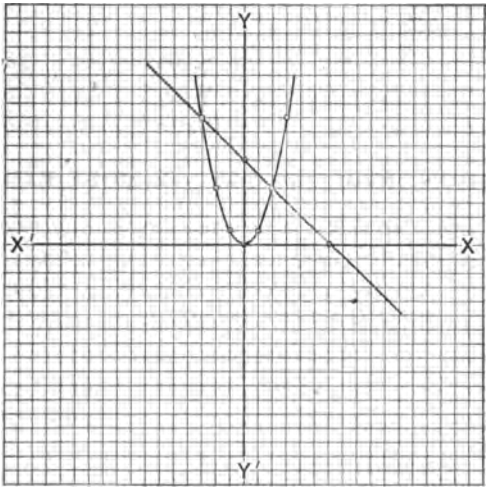


FIG. IV.

The graph of any equation of the form $ax^2 + bx + c = 0$ as shown, is one of the curves produced by conic sections and is called a *parabola*.

Note.—When the roots of an equation are fractional, the graph will not intersect the X axis at a point of division and the values of the roots must be estimated approximately. In such cases, the graph should be drawn as accurately as possible.

A modification of the foregoing method is sometimes used.

4. Solve $x^2 + x - 6 = 0$.

Let $\qquad y = x^2$ (1)

Then $y + x - 6 = 0$ (2)

Using these two equations as a simultaneous system, the graph of equation (1) is a parabola, and that of equation (2) a straight line, Fig. IV.

The points of intersection of these two graphs, will locate the roots. The straight line cuts the parabola when $x = +2$ and when $x = -3$. Hence these are the required roots.

The advantage of this method lies in the fact that the same parabola $y = x^2$ may be used for any number of equations.

EXERCISE 100

Find, by graphs, the roots of:

1. $x^2 - x - 6 = 0$.
2. $x^2 - 3x - 4 = 0$.
3. $x^2 + 5x + 4 = 0$.
4. $x^2 - 8x + 12 = 0$.
5. $x^2 - 6x + 9 = 0$.
6. $2x^2 - 5x + 2 = 0$.
7. $8x^2 - 2x - 3 = 0$.
8. $4x^2 + 4x - 15 = 0$.
9. $2x^2 + 5x - 25 = 0$.
10. $6x^2 + 10x + 6 = 0$.

EQUATIONS INVOLVING TWO OR MORE RADICALS

141. In order to solve equations containing two or more radicals, practically the same methods are used as in the case of first degree equations involving radicals. The equation must be cleared of radicals, usually by raising the equation to the necessary power. If the equation contains two or more radicals, involution may be necessary two or more times to rationalize the equation.

The student should note the importance of simplifying the equation before applying involution.

1. Solve $\sqrt{x+6} - \sqrt{x+1} = \sqrt{x-2}$.

Squaring, $x + 6 - 2\sqrt{(x+6)(x+1)} + x + 1 = x - 2$

Collecting, $x + 9 = 2\sqrt{(x+6)(x+1)}$

Squaring, $x^2 + 18x + 81 = 4(x+6)(x+1)$

$x^2 + 18x + 81 = 4x^2 + 28x + 24$

Transposing and collecting, $3x^2 + 10x = 57$

Mult. by coeff. of x^2, $9x^2 + 30x = 171$

Add, to both sides, the square of half the original coefficient of x,

$9x^2 + 30x + 25 = 196$

Extract square root, $3x + 5 = \pm 14$

Transpose, $3x = \pm 14 - 5$

$\therefore\ 3x = 9$ or -19

And $x = 3$ or $-\frac{19}{3}$

Verify the solution, by substituting the values of x in the original equation.

If $x = 3$,
$$\sqrt{3+6} - \sqrt{3+1} = \sqrt{3-2}$$
$$\sqrt{9} - \sqrt{4} = \sqrt{1}$$
$$3 - 2 = 1$$
$$1 = 1$$

Hence 3 is a root of this equation.

If $x = -\frac{19}{3}$,
$$\sqrt{-\tfrac{19}{3}+6} - \sqrt{-\tfrac{19}{3}+1} = \sqrt{-\tfrac{19}{3}-2}$$
$$\sqrt{-\tfrac{1}{3}} - \sqrt{-\tfrac{16}{3}} = \sqrt{-\tfrac{25}{3}}$$
$$\tfrac{1}{3}\sqrt{-3} - \tfrac{4}{3}\sqrt{-3} = \tfrac{5}{3}\sqrt{-3}$$

As this equation does not reduce to an identity, $-\frac{19}{3}$ is not a root of this equation.

In the solution of radical equations, involution may introduce factors into the equation that will produce values of x that do not satisfy the original equation. Hence the values found should always be substituted in the original equation to see whether they are roots. Values that do not satisfy the equation should be discarded.

2. Solve $\sqrt{x^2+5} - \dfrac{3}{\sqrt{x^2+5}} = x.$

Clear of fractions, $x^2 + 5 - 3 = x\sqrt{x^2+5}$
$$x^2 + 2 = x\sqrt{x^2+5}$$

Squaring, $x^4 + 4x^2 + 4 = x^4 + 5x^2$

Transpose and collect, $x^2 = 4$
$$x = \pm 2$$

Verifying,

$x = 2$,
$$\sqrt{4+5} - \frac{3}{\sqrt{4+5}} = 2$$
$$\sqrt{9} - \frac{3}{\sqrt{9}} = 2$$
$$3 - 1 = 2$$
$$2 = 2$$

Hence $+2$ is a root of the equation.

$x = -2$,
$$\sqrt{4+5} - \frac{3}{\sqrt{4+5}} = -2$$
$$\sqrt{9} - \frac{3}{\sqrt{9}} = -2$$
$$3 - 1 = -2$$
$$2 = -2$$

Hence -2 is not a root.

EXERCISE 101

Solve and verify:

1. $\sqrt{x+4} - \sqrt{x-1} = \sqrt{x-4}$.
2. $2\sqrt{3x} - x = 3$.
3. $(x+3)(x+1) = \sqrt{13 + x(x^3+2)} + 4x$.
4. $\sqrt{4 + x\sqrt{3x^2+4x}} = x + 2$.
5. $\sqrt{3x+10} - \sqrt{x+7} = \frac{1}{2}\sqrt{2x}$.
6. $\dfrac{\sqrt{x}+17}{2\sqrt{x}+4} - \dfrac{\sqrt{x}+11}{2\sqrt{x}+1} = 0$.
7. $\sqrt{3x+4} - \sqrt{8-x} = \sqrt{2x-4}$.
8. $3x - \sqrt{4x+1} = x + 1$.
9. $\sqrt{4x+5} + \sqrt{2x-6} - \sqrt{8x+9} = 0$.
10. $\sqrt{3x-2} - \sqrt{x-2} = \sqrt{\frac{2}{3}x}$.
11. $\sqrt{x^2+12} - \dfrac{8}{\sqrt{x^2+12}} = x$.
12. $\dfrac{1}{\sqrt{x^2+1} - x} - \dfrac{1}{\sqrt{x^2+1} + x} = 14$.
13. $\sqrt{x^2+a^2} = \dfrac{2a^2}{\sqrt{x^2+a^2}} - x$.
14. $\dfrac{5x - 4\sqrt{x^2-9}}{x - \sqrt{x^2-9}} = x + \sqrt{x^2-9}$.
15. $\sqrt{\dfrac{x-3}{x+3}} + \sqrt{\dfrac{x+3}{x-3}} = 9$.
16. $\sqrt{x-ab} + \sqrt{x}\sqrt{ab-1} = \sqrt{ab}\sqrt{x-1}$.

EQUATIONS IN THE QUADRATIC FORM

142. A complete quadratic equation contains the second and first power of the unknown number, that is, one exponent is twice the other. Hence, any equation that contains only two powers of the unknown number is said to be in the **quadratic form,** if one of the exponents is twice the other. Such equations may be solved by the methods used in solving quadratics.

1. Solve $4x^4 - 25x^2 = -36$.

Completing the square by adding to each side the square of the quotient obtained by dividing the second term by twice the square root of the first term,

$$4x^4 - 25x^2 + (\tfrac{25}{4})^2 = -36 + \tfrac{625}{16} = \tfrac{49}{16}$$

Extracting the square root,

$$2x^2 - \tfrac{25}{4} = \pm\tfrac{7}{4}$$

Transposing, $2x^2 = \tfrac{25}{4} \pm \tfrac{7}{4}$

$$2x^2 = 8 \text{ or } \tfrac{18}{4}$$

Dividing by 2, $x^2 = 4 \text{ or } \tfrac{9}{4}$

Extracting the square root, $x = \pm 2 \text{ or } \pm\tfrac{3}{2}$

Hence this equation of the fourth degree has the four roots, $+2$, -2, $+\tfrac{3}{2}$ and $-\tfrac{3}{2}$

2. Solve $6x^{\frac{1}{2}} + 6x^{-\frac{1}{2}} = 13$.

Changing the sign of the exponent,

$$6x^{\frac{1}{2}} + \frac{6}{x^{\frac{1}{2}}} = 13$$

Clearing of fractions,

$$6x + 6 = 13x^{\frac{1}{2}}$$

Transposing, $6x - 13x^{\frac{1}{2}} = -6$, an equation in the quadratic form.

Multiplying by 6, $36x - 78x^{\frac{1}{2}} = -36$

Completing the square, $36x - 78x^{\frac{1}{2}} + (\tfrac{13}{2})^2 = -36 + \tfrac{169}{4} = \tfrac{25}{4}$

Extracting the square root, $6x^{\frac{1}{2}} - \tfrac{13}{2} = \pm\tfrac{5}{2}$

Transposing, $6x^{\frac{1}{2}} = \tfrac{13}{2} \pm \tfrac{5}{2}$

$$\therefore\ 6x^{\frac{1}{2}} = 9 \text{ or } 4$$

Dividing by 6, $x^{\frac{1}{2}} = \tfrac{3}{2} \text{ or } \tfrac{2}{3}$

Squaring, $x = \tfrac{9}{4} \text{ or } \tfrac{4}{9}$

3. Solve $x^2 - 7x + \sqrt{x^2 - 7x - 14} = 20$.

Adding -14 to each side of the equation,

$$x^2 - 7x - 14 + \sqrt{x^2 - 7x - 14} = 6$$

let $y = \sqrt{x^2 - 7x - 14}$, then $y^2 = x^2 - 7x - 14$

Substituting these values,

$$y^2 + y = 6$$

Transposing, $y^2 + y - 6 = 0$

Factoring, $(y + 3)(y - 2) = 0$

Hence $y + 3 = 0$ and $y - 2 = 0$

$$y = -3 \qquad y = 2$$

Substituting the value of y,

$$\sqrt{x^2 - 7x - 14} = -3 \text{ or } 2$$

Squaring, $x^2 - 7x - 14 = 9 \text{ or } 4$

Solving these two quadratic equations,

$$x^2 - 7x - 14 = 9 \qquad\qquad x^2 - 7x - 14 = 4$$

$$x^2 - 7x = 23 \qquad\qquad x^2 - 7x - 18 = 0$$

$$x^2 - 7x + (\tfrac{7}{2})^2 = 23 + \tfrac{49}{4} = \tfrac{141}{4} \qquad (x - 9)(x + 2) = 0$$

$$x - \tfrac{7}{2} = \pm\tfrac{1}{2}\sqrt{141} \qquad x - 9 = 0 \quad x + 2 = 0$$

$$x = \tfrac{7}{2} \pm \tfrac{1}{2}\sqrt{141} \qquad x = 9 \qquad x = -2$$

4. Solve $4x^4 + 12x^3 - 11x^2 - 30x + 16 = 0$.

Extract the square root of the left side,

$$\begin{array}{r|l}
 & 4x^4 + 12x^3 - 11x^2 - 30x + 16 \;)\underline{2x^2 + 3x - 5} \\
 & 4x^4 \\ \hline
4x^2 + 3x & 12x^3 - 11x^2 \\
 & 12x^3 + 9x^2 \\ \hline
4x^2 + 6x - 5 & -20x^2 - 30x + 16 \\
 & -20x^2 - 30x + 25 \\ \hline
 & \qquad\qquad\qquad -9
\end{array}$$

The remainder, -9, shows that the expression lacks 9 of being a perfect square.

Add 9 to each side of the equation,

$$4x^4 + 12x^3 - 11x^2 - 30x + 25 = 9$$

Extracting the square root,

$$2x^2 + 3x - 5 = \pm 3$$

This gives us two quadratic equations to solve,

$$\begin{array}{rcl}
2x^2 + 3x - 5 = 3 & \text{and} & 2x^2 + 3x - 5 = -3 \\
2x^2 + 3x = 8 & & 2x^2 + 3x - 2 = 0 \\
x^2 + \frac{3}{2}x = 4 & & (2x - 1)(x + 2) = 0 \\
x^2 + \frac{3}{2}x + (\frac{3}{4})^2 = 4 + \frac{9}{16} = \frac{73}{16} & & 2x - 1 = 0 \quad x + 2 = 0 \\
x + \frac{3}{4} = \pm\frac{1}{4}\sqrt{73} & & 2x = 1 \qquad x = -2 \\
x = -\frac{3}{4} \pm \frac{1}{4}\sqrt{73} & & x = \frac{1}{2}
\end{array}$$

EXERCISE 102

Solve:

1. $x^4 - 4x^2 - 45 = 0$.
2. $x^4 - 13x^2 + 36 = 0$
3. $5x^4 + 6x^2 = 11$.
4. $x^6 - 7x^3 - 8 = 0$.
5. $x - 5x^{\frac{1}{2}} + 6 = 0$.
6. $x + 3\sqrt{x} = 28$.
7. $x^{\frac{1}{2}} + 2x^{\frac{1}{4}} = 3$.
8. $x^{\frac{1}{3}} - x^{\frac{1}{6}} = 6$.
9. $x^{\frac{2}{3}} - 10x^{\frac{1}{3}} + 9 = 0$.
10. $x^{\frac{1}{2}} + 2x^{-\frac{1}{2}} = 3$.
11. $(x + 2)^2 + 2(x + 2) = 3$.
12. $(x^2 - 1)^2 + 24 = 11(x^2 - 1)$.
13. $x + 2\sqrt{x - 3} = 6$.
14. $x^2 - 3x + 4 + 2\sqrt{x^2 - 3x + 6} = 14$.
15. $x^2 - x - \sqrt{x^2 - x + 4} = 8$.
16. $x^2 + 5\sqrt{x^2 - 5x + 1} = 5x + 49$.
17. $\left(x + \frac{1}{x}\right)^2 + 2\left(x + \frac{1}{x}\right) = 8$.
18. $x^4 + 4x^3 - 8x - 5 = 0$.
19. $x^4 + 4x^3 - 2x^2 - 12x + 5 = 0$.
20. $4x^4 + 4x^3 - 5x^2 - 6x + 8 = 0$.

GIROLAMO CARDANO

(From an Old Engraving)

Born at Pavia, 1501. Died at Rome, 1576.

Jerome Cardan was one of the greatest mathematicians of his time. His name is usually applied to the general solution of the cubic equation, although this was really the work of *Tartaglia* in 1541.

Cardan was astrologer, physician, philosopher and mathematician.

HIGHER EQUATIONS SOLVED BY FACTORING

143. Equations of a higher degree than the second can often be solved by factoring and placing each factor equal to zero, as previously explained.

1. Solve, $2x^3 - x^2 - 15x = 0$.

Factoring, $x(2x^2 - x - 15) = 0$.

$x(2x + 5)(x - 3) = 0$.

Place each factor equal to zero,

$$x = 0 \qquad \begin{aligned} 2x + 5 &= 0 \\ 2x &= -5 \\ x &= -\tfrac{5}{2} \end{aligned} \qquad \begin{aligned} x - 3 &= 0 \\ x &= 3 \end{aligned}$$

Hence, the three roots are, 0, $-\frac{5}{2}$ and 3.

2. Find the three cube roots of 1.

Let $x =$ the required cube root.
Then $x^3 = 1$
And $x^3 - 1 = 0$
Factoring, $(x - 1)(x^2 + x + 1) = 0$
Place each factor equal to zero.

$$\begin{aligned} x - 1 &= 0 \\ x &= 1 \end{aligned} \qquad \begin{aligned} x^2 + x + 1 &= 0 \\ x^2 + x &= -1 \\ x + x + \tfrac{1}{4} &= -1 + \tfrac{1}{4} = -\tfrac{3}{4} \\ x + \tfrac{1}{2} &= \pm\tfrac{1}{2}\sqrt{-3} \\ x &= -\tfrac{1}{2} \pm \tfrac{1}{2}\sqrt{-3} \end{aligned}$$

Note that any method of factoring may be used, also that the degree of the equation indicates the number of roots.

EXERCISE 103

Solve by factoring:

1. $x^4 - 13x + 36 = 0$.
2. $4x^4 - 25x^2 + 36 = 0$.
3. $36x^4 - 97x^2 + 36 = 0$.
4. $x^3 - 5x^2 + 4x = 0$.
5. $x^3 - 8 = 0$.
6. $x^4 - 16 = 0$.
7. $x^3 + 4x^2 + 3x + 12 = 0$.
8. $2x^3 - 3x^2 - 4x + 6 = 0$.
9. $x^4 + x^2 + 1 = 0$.
10. $x^3 + 8x + 3 = 0$.
11. $x^3 - 3x^2 + 4x - 2 = 0$.
12. $4x^3 - 12x^2 + 9x - 1 = 0$.
13. $4x^4 - 9x^2 + 6x - 1 = 0$.

14. $x^3 - 8 - 6x + 3x^2 = 0.$
15. $x^3 - 8 - (x^2 - 4) - (x - 2)^2 = 0.$
16. $x^6 - 64 = 0.$
17. $x^3 - 1 - 13(x - 1) = 0.$
18. $3x^3 - 3x^2 + 2(x^2 + 3x - 4) = 0.$

PROBLEMS INVOLVING QUADRATICS

144. Since a quadratic equation has two roots, problems involving quadratics seem to have two solutions. If both roots are positive integers, there will usually be two solutions of the problem. If the roots are fractional or negative, they may or may not give solutions, depending upon the conditions stated in the problem. If the roots are imaginary, the problem is impossible.

1. The difference of two numbers is 4 and their product is 12. Find the numbers.

Let $x =$ the smaller number
Then $x + 4 =$ the larger number
Therefore $x(x + 4) = 12$
$x^2 + 4x = 12$
$x^2 + 4x - 12 = 0$
$(x + 6)(x - 2) = 0$
$x + 6 = 0 \qquad x - 2 = 0$
$x = -6 \qquad x = 2$

When $x = 2$, $x + 4 = 6$, hence the two numbers are 2 and 6. When $x = -6$, $x + 4 = -2$, hence the two numbers may also be 2 and -6. Evidently both solutions satisfy the conditions stated in the problem.

2. Find two consecutive numbers, the sum of whose squares is 145.

Let $x =$ the smaller number
Then $x + 1 =$ the larger number
$(x + 1)^2 + x^2 = 145$
$x^2 + 2x + 1 + x^2 = 145$
$2x^2 + 2x = 144$
$x^2 + x = 72$
$x^2 + x - 72 = 0$
$(x + 9)(x - 8) = 0$
$x + 9 = 0 \qquad x - 8 = 0$
$x = -9 \qquad x = 8$

When $x = 8$, $x + 1 = 9$ and the numbers are 8 and 9.

The negative value of x cannot be used here as consecutive numbers are understood to be integers which follow each other in the common scale 1, 2, 3, 4. . . . .

3. A club ordered a dinner that cost $150.00. Ten of the members failed to attend the dinner and each of the other members had to pay 75 cents more. How many members were there in the club?

Let x = the number of members in the club,

Then $\frac{150}{x}$ = the number of dollars each would have paid.

As 10 members were absent,

$x - 10$ = the number of members who attended.

And $\frac{150}{x - 10}$ = the number of dollars each paid.

But as each had to pay 75¢ more.

$$\frac{150}{x-10} - \frac{150}{x} = \frac{3}{4}$$

Solving this equation,

$$\begin{aligned} 600x - 600x\,(x - 10) &= 3x(x - 10) \\ 600x - 600x + 6000 &= 3x^2 - 30x \\ 3x^2 - 30x &= 6000 \\ x^2 - 10x &= 2000 \\ x^2 - 10x - 2000 &= 0 \\ (x - 50)(x + 40) &= 0 \end{aligned}$$

$$x - 50 = 0 \qquad x + 40 = 0$$

$$x = 50 \qquad x = -40.$$

The negative root is evidently inadmissible.

Therefore the number of members is 50.

4. A clothing merchant sold a suit for $39 and gained as many percent as the suit cost him in dollars. What was the cost of the suit?

Let x = the number of dollars in the cost.

Then x = the number of percent gain.

But $\frac{x^2}{100}$ = the number of dollars he gained.

Therefore $$x + \frac{x^2}{100} = 39$$

$$\begin{aligned} 100x + x^2 &= 3900 \\ x^2 + 100x + 2500 &= 3900 + 2500 = 6400 \\ x + 50 &= \pm 80 \\ x &= \pm 80 - 50 \\ x &= 30 \text{ or } -130 \end{aligned}$$

The negative root is not applicable.

Hence the suit cost $30.

5. A circular flower bed is surrounded by a strip of grass having a uniform width of 2 feet. The area of the grass strip is $\frac{24}{25}$ of the area of the flower bed. Find the radius of the flower bed.

Let x = the number of feet in the radius of the flower bed.

Then $x + 2$ = the number of feet in the radius of the bed and strip.

As the area of a circle is found approximately by multiplying the square of the radius by 3.1416+, usually represented by π.

$\pi(x+2)^2$ = the number of square feet in the area of both strip and bed.

And πx^2 = the number of square feet in the area of the flower bed.

$$
\begin{aligned}
\text{Then}\qquad \pi(x+2)^2 - \pi x^2 &= \tfrac{24}{25}\pi x^2\\
\text{Dividing by } \pi,\qquad (x+2)^2 - x^2 &= \tfrac{24}{25}x^2\\
\text{Solving, } 25(x^2+4x+4) - 25x^2 &= 24x^2\\
25x^2 + 100x + 100 - 25x^2 &= 24x^2\\
24x^2 - 100x &= 100\\
6x^2 - 25x &= 25\\
36x^2 - 150x &= 150\\
36x^2 - 150x + (\tfrac{25}{2})^2 &= 150 + \tfrac{625}{4} = \tfrac{1225}{4}\\
6x - \tfrac{25}{2} &= \pm\tfrac{35}{2}\\
6x &= \tfrac{25}{2} \pm \tfrac{35}{2}\\
6x &= 30 \text{ or } -5\\
x &= 5 \text{ or } -\tfrac{5}{6}
\end{aligned}
$$

The positive root only can be used here.

Hence the radius of the bed is 5 feet.

EXERCISE 104

1. The sum of two numbers is 10 and their product 21. Find the numbers.

2. Divide 18 into two parts whose product is 65.

3. Find two consecutive integers, the sum of whose squares is 265.

4. Find two consecutive odd numbers, the difference of whose cubes is 98.

5. The sum of the squares of two consecutive numbers exceeds the square of the next number by 45. Find the numbers.

6. The sum of a number and 5 times its reciprocal is 6. Find the number.

7. A rectangular field containing 1 acre is 6 rods longer than it is wide. Find the dimensions of the field.

8. The length of a rectangular floor is 4 feet more than its width. If each dimension were increased by 3 feet, its area would be increased by 9 square yards. Find its dimensions.

9. The perimeter (sum of the sides) of a rectangular field is 120 rods and its area 600 square rods. Find its dimensions.

10. The denominator of a certain fraction exceeds the numerator by 1. If 2 is added to both numerator and denominator, the value of the fraction is increased by $\frac{1}{12}$. Find the fraction.

11. A man agreed to work a certain number of days for $54. Had he worked 9 days more for the same amount he would have earned $1 less per day. Find the number of days he worked.

12. A tennis club rented a plot of ground for $100. Five of the members left the club and each of the remaining members had to pay $1 more. Find the original number of members.

13. A merchant bought a number of chairs for $108. He sold all but 6 for $112.50, with a profit of 75¢ on each chair sold. How many chairs did he buy?

14. A train, making a run of 90 miles, was delayed 12 minutes at the start. To arrive on time, it increased its usual speed by 5 miles per hour. Find the usual rate per hour.

15. A boat's crew can row 6 miles down a river and back again in 2 hours 40 minutes. The rate of the crew in still water is twice the rate of the current. Find the rate of the crew in still water and the rate of the current.

16. A water tank can be filled by two pipes running together in 18 minutes. The larger pipe can fill the tank in 15 minutes less time than the smaller pipe. Find the time required by each pipe alone to fill the tank.

17. Find the price of eggs, if 2 more for 40¢ lowers the price 12¢ per dozen.

18. A number consists of two digits. The units' digit is equal to the square of half the tens' digit. If 27 is added to the number, the order of the digits will be reversed. Find the number.

19. A dealer bought hardware, listed at \$50, for \$24, receiving two successive discounts. Find the discounts in per cent, if the first was twice the second.

20. If a carriage wheel with a circumference of 11 ft. took $\frac{1}{4}$ of a second longer to revolve, the rate of the carriage would be decreased 5 miles per hour. Find the rate of the carriage.

REVIEW EXERCISE XVII

Solve:

1. $(x^2 - x - 6)(x^2 - x - 20) = 0.$
2. $x^5 - 9x^3 + x^2 - 9 = 0.$
3. $2x^2 - 3x - 21 = 2x\sqrt{x^2 - 3x + 4}.$
4. $\dfrac{x^2}{3m - 2a} - \dfrac{m^2 - 4a^2}{4a - 6m} = \dfrac{x}{2}.$
5. $x^4 - 4x^3 - 10x^2 + 28x - 15 = 0.$
6. $\sqrt{a - x} + \sqrt{b - x} = \sqrt{a + b - 2x}.$
7. $(x - a)^{\frac{1}{2}} + 2\sqrt{b}\,(x - a)^{\frac{1}{4}} - 3b = 0.$
8. $x^{-4} - 21x^{-2} = -108.$
9. $x^2 + 7 + \sqrt{x^2 + 7} = 20.$
10. $\dfrac{\sqrt{x} + 25}{\sqrt{x} + 7} = \dfrac{\sqrt{x} + 4}{\sqrt{x} + 1}.$
11. $\dfrac{x}{x^2 + 1} + \dfrac{x^2 + 1}{x} = \dfrac{5}{2}.$
12. $\sqrt{(a + b)x - 4ab} = x - 2b.$
13. $32x^5 + \dfrac{1}{x^5} = -33.$
14. $\left(x - \dfrac{6}{x}\right)^2 + 4x - \dfrac{24}{x} = 5.$
15. $x^3 - 1000 = 0.$
16. $\dfrac{1}{x^3} - 7x^{-\frac{3}{2}} = 8.$

17. $3x(3 - x) = 11 - 4\sqrt{x^2 - 3x + 5}$.

18. $2x^2 - 2x + 2\sqrt{2x^2 - 7x + 6} = 5x - 6$.

19. $x^2 - x + 3\sqrt{2x^2 - 3x + 2} = \frac{x}{2} + 7$.

20. $\sqrt{2a^2x^2 - 6ax - a^2 + 5} = a - 1$.

Graphically, locate the roots of:

21. $x^2 - 2x - 2 = 0$.

22. $x^2 - 3x + 5 = 0$.

23. $x^2 + 2x - 3 = 0$.

24. $x^2 + 2x - 8 = 0$.

25. $x^2 + 3x - 4 = 0$.

26. $2x^2 - 7x - 49 = 0$.

27. If a certain number is divided by 8, the result will be the same as if 16 were divided by the number and $3\frac{1}{2}$ added to the quotient. Find the number.

28. A man buys a certain number of pounds of meat for $8 If he had paid 2 cents per pound more, he would have received 20 pounds less for his money. How many pounds did he buy?

29. Some boys laid out basketball grounds, 30 feet greater in length than in width, but to change the area to the prescribed limit of 3500 square feet, they reduced the length 10 feet. How much too large had they laid out the grounds?

30. If an automobile traveled 3 miles an hour faster, it would require 2 hours less time to go 120 miles. What is the present rate of the automobile in miles per hour?

31. A certain number of eggs cost a dollar. If there had been 10 more eggs for a dollar, they would have cost 6 cents per dozen less. Find the cost per dozen.

32. Sold wheat for $56 and gained as many per cent as the wheat cost in dollars. Find the cost of the wheat.

33. A man travels 24 miles by an accommodation train and returns by an express train which runs 10 miles an hour faster. Find the rates of the two trains in miles per hour if the time taken by the two trains was one hour and twenty-four minutes.

34. The product of three consecutive numbers is equal to three times the middle number. Find the numbers.

35. Two square floors are paved with stones 1 foot square. The length of the side of one floor is 12 feet more than that of the other. The number of stones in the two floors is 2120. Find the size of each floor.

36. A man sold a horse for $35 and half as much as he gave for it, and gained $10 thereby. What did he pay for it?

37. If a carriage wheel $14\frac{2}{3}$ feet in circumference takes one second more to revolve, the rate of the carriage would be $2\frac{2}{3}$ miles less per hour. How fast is the carriage travelling?

CHAPTER XVIII

SIMULTANEOUS QUADRATIC EQUATIONS

145. Two simultaneous quadratic equations, involving two unknown numbers, are solved by various methods, depending upon the form of the equations.

I. When one of the equations is a simple equation.

$$\text{Solve } \begin{cases} x^2 + xy = 12 \ (1) \\ x - y = 2 \ (2) \end{cases}$$

From equation ②, $y = x - 2$

Substitute this value for y in equation ①,

$$x^2 + x(x - 2) = 12$$

Solving,

$$\begin{aligned} x^2 + x^2 - 2x &= 12 \\ 2x^2 - 2x &= 12 \\ x^2 - x &= 6 \\ x^2 - x - 6 &= 0 \\ (x - 3)(x + 2) &= 0 \end{aligned}$$

$$x - 3 = 0 \qquad x + 2 = 0$$

$$x = 3 \qquad x = -2$$

Since $y = x - 2$, $y = 3 - 2$ or $-2 - 2$

Hence $y = 1$ or -4

These values must be properly grouped, showing the corresponding values of x and y.

$$\text{Thus, } \begin{cases} x = 3 \\ y = 1 \end{cases} \qquad \begin{cases} x = -2 \\ y = -4 \end{cases}$$

The student should note that if the proper pair of values is substituted in the original equations, the equations will be satisfied.

EXERCISE 105

Solve:

1. $\begin{cases} x^2 + y^2 = 10 \\ x + 2y = 7. \end{cases}$

2. $\begin{cases} x^2 - y^2 = 16 \\ 3x = y + 12. \end{cases}$

3. $\begin{cases} x^2 + xy = 33 \\ y - 3x = 1. \end{cases}$

4. $\begin{cases} 3x^2 = 4y^2 + 8 \\ 5x = 4y + 10. \end{cases}$

5. $\begin{cases} x - y = -2 \\ 3xy - 10x = y. \end{cases}$

6. $\begin{cases} x^2 - 3y^2 = 13 \\ x - 2y = 1. \end{cases}$

7. $\begin{cases} x^2 - 2xy + 3y = x + 3 \\ x = 4 - y. \end{cases}$

8. $\begin{cases} x^2 - x - 4y^2 = 2y \\ 7 - 4y = x. \end{cases}$

9. $\begin{cases} x^2 = 7 - 3xy \\ x = 5 - 2y. \end{cases}$

12. $\begin{cases} x^2 + 3 = y^2 + 3xy \\ 3x = 5 - 2y. \end{cases}$

10. $\begin{cases} 2x + 3y = 12 \\ xy = 6. \end{cases}$

13. $\begin{cases} x + 2 = xy \\ 9 - 2y = 3x. \end{cases}$

11. $\begin{cases} y^2 - 4 = 3xy \\ x + 2y = -1. \end{cases}$

14. $\begin{cases} 2x + 3y = 5 \\ x^2 + xy = 2. \end{cases}$

15. $\begin{cases} 3x + 2y = -7 \\ 2x^2 + 3y^2 - y = 12 - 4x. \end{cases}$

II. When the left sides of the two equations are homogeneous and of the second degree.

146. Equations are **homogeneous** when all their terms are of the same degree with respect to the unknown numbers. Thus, $2x^2$, $3xy$, y^2 are terms of the second degree and the equation $2x^2 + 3y^2 - 2xy = 0$, is homogeneous.

The equation $2x^2 - 3xy + y^2 = 12$, is homogeneous on one side only, or it may be said to be homogeneous in its unknown terms. Such equations are most easily solved by the following method.

1. Solve $\begin{cases} -4x^2 + xy + y^2 = 18 \\ y^2 - xy + 9 = 3x^2 \end{cases}$

Rearranging the terms,

$$4x^2 - xy - y^2 = -18 \quad ①$$
$$3x^2 + xy - y^2 = 9 \quad ②$$

Let $y = vx$.

Substituting in ①, $4x^2 - vx^2 - v^2x^2 = -18$

$$(4 - v - v^2)x^2 = -18$$

$$x^2 = \frac{-18}{4 - v - v^2} = \frac{18}{v^2 + v - 4} \quad ③$$

Substituting in ②, $3x^2 + vx^2 - v^2x^2 = 9$

$$(3 + v - v^2)x^2 = 9$$

$$x^2 = \frac{9}{3 + v - v^2} \quad ④$$

Equating the values of x^2 in ③ and ④,

$$\frac{18}{v^2+v-4}=\frac{9}{3+v-v^2}.$$

Divide by 9, $$\frac{2}{v^2+v-4}=\frac{1}{3+v-v^2}.$$

Clearing of fractions, $2(3+v-v^2)=v^2+v-4$

$$6+2v-2v^2=v^2+v-4$$
$$v-3v^2=-10$$
$$3v^2-v=10$$
$$v^2-\tfrac{1}{3}v=\tfrac{10}{3}$$
$$v^2-\tfrac{1}{3}v+(\tfrac{1}{6})^2=\tfrac{10}{3}+\tfrac{1}{36}=\tfrac{121}{36}$$
$$v-\tfrac{1}{6}=\pm\tfrac{11}{6}$$
$$v=\frac{1\pm 11}{6}$$
$$v=2 \text{ or } -\tfrac{5}{3}$$

Substituting these values in ③,

When $v=2$,

$$x^2=\frac{18}{4+2-4}=\frac{18}{2}=9.$$
$$x=\pm 3$$

And $y=\pm 6$

When $v=-\tfrac{5}{3}$,

$$x^2=\frac{18}{\frac{25}{9}-\frac{5}{3}-4}=\frac{18}{-\frac{26}{9}}=-\frac{81}{13}$$
$$x=\pm 9\sqrt{-\tfrac{1}{13}}=\pm\tfrac{9}{13}\sqrt{-13}$$
$$y=\mp\tfrac{15}{13}\sqrt{-13}.$$

EXERCISE 106

Solve:

1. $\begin{cases} 2x^2-4xy+3y^2=17 \\ x^2-y^2=16. \end{cases}$

2. $\begin{cases} x^2+xy+y^2=39 \\ x^2+3xy+2y^2=63. \end{cases}$

3. $\begin{cases} x^2+3y^2=13 \\ xy+2y^2=10. \end{cases}$

4. $\begin{cases} xy+3x^2=20 \\ y^2+8=3xy. \end{cases}$

5. $\begin{cases} 2x^2+xy=5 \\ xy+y^2=12. \end{cases}$

6. $\begin{cases} x^2+8=xy \\ 2x^2+y^2=44. \end{cases}$

7. $\begin{cases} y^2-xy-x^2=20 \\ 2x^2-3xy+y^2=8. \end{cases}$

8. $\begin{cases} x^2-xy+y^2=7 \\ 2x^2+y^2=9. \end{cases}$

9. $\begin{cases} 2x^2 - 3xy + 2y^2 = 20 \\ y^2 - x^2 = 15. \end{cases}$

10. $\begin{cases} 3x^2 - 5xy + y^2 = 1 \\ x^2 + xy + 3y^2 = 3. \end{cases}$

11. $\begin{cases} x^2 - xy + y^2 = 7 \\ x^2 - 2xy = -5. \end{cases}$

12. $\begin{cases} 4x^2 - 5y^2 = -44 \\ 5x^2 - 4y^2 = -19. \end{cases}$

13. $\begin{cases} x^2 + 3xy + 2y^2 = 20 \\ 4x^2 + 5y^2 = 41. \end{cases}$

14. $\begin{cases} 3xy + 5y^2 = 104 \\ x^2 = 2xy - 12. \end{cases}$

15. $\begin{cases} x^2 - 11 = 10y^2 + 7xy \\ x^2 - 15 = 12y^2 + 9xy. \end{cases}$

III. When the two equations are symmetrical with respect to the two unknown numbers.

147. Algebraic expressions are said to be **symmetrical** with respect to two unknowns, when the two unknowns are similarly involved. In all symmetrical expressions, the two unknowns may be interchanged without altering the form of the expression.

As,

$$x^2 - xy + y^2,\ 2x^2 + 3xy + 2y^2,\ x^3 + 3x^2y + 3xy^2 + y^3.$$

The direct object of this method of solving symmetrical equations is to obtain the values of $x + y$ and $x - y$, whence, by combining these two simple equations, we obtain the values of x and y.

1. Solve $\begin{cases} x^2 + y^2 = 25 \ (1) \\ xy = 12 \ (2) \end{cases}$

Multiply ② by 2,

$$2xy = 24 \quad ③$$

Add ① and ③, $x^2 + 2xy + y^2 = 49$ ④

Extract square root, $x + y = \pm 7$ ⑤

To obtain the value of $x - y$,

Subtract ③ from ①, $x^2 - 2xy + y^2 = 1$

Extract square root, $x - y = \pm 1$ ⑥

Combine ⑤ and ⑥, $x + y = \pm 7$ ⑤

$\underline{x - y = \pm 1}$ ⑥

By addition, $2x = \pm 8$

$x = \pm 4$

By subtraction, $2y = \pm 6$

$y = \pm 3$

Hence $\begin{cases} x = 4 \\ y = 3 \end{cases}$ $\begin{cases} x = -4 \\ y = -3 \end{cases}$

2. Solve $\begin{cases} x^4 + y^4 = 17 \ (1) \\ x + y = 3 \ (2) \end{cases}$

Raise ② to the fourth power,

$x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4 = 81$ ③

Add ① and ③, $\underline{x^4 \qquad\qquad\qquad\qquad + y^4 = 17}$

$2x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + 2y^4 = 98$ ④

Divide by 2, $x^4 + 2x^3y + 3x^2y^2 + 2xy^3 + y^4 = 49$ ⑤

Extract the square root, $x^2 + xy + y^2 = \pm 7$ ⑥

As the value of $x + y$ is given in (2), we must find the value of $x - y$.

Square ②, $x^2 + 2xy + y^2 = 9$

Subtract ⑥, $\underline{x^2 + xy + y^2 = \pm 7}$

$xy = 2 \text{ or } 16$

and $3xy = 6 \text{ or } 48$ ⑦

Subtract ⑦ from ⑥, $x^2 - 2xy + y^2 = 1 \text{ or } -55$

Extract the square root, $x - y = \pm 1 \text{ or } \pm\sqrt{-55}$ ⑧

Combining ② and ⑧, $x + y = 3$

$\underline{x - y = \pm 1 \text{ or } \pm\sqrt{-55}}$

By addition, $2x = 4 \text{ or } 2 \text{ or } 3 \pm \sqrt{-55}$

and $x = 2 \text{ or } 1 \text{ or } \dfrac{3 \pm \sqrt{-55}}{2}$

By subtraction, $2y = 2 \text{ or } 4 \text{ or } 3 \mp \sqrt{-55}$

$y = 1 \text{ or } 2 \text{ or } \dfrac{3 \mp \sqrt{-55}}{2}$

When the signs are used in their proper order, the corresponding values of x and y are found in their proper order, as above

$$\left.\begin{matrix} x = 2 \\ y = 1 \end{matrix}\right\} \left.\begin{matrix} \text{or } 1 \\ \text{or } 2 \end{matrix}\right\} \left.\begin{matrix} \text{or } \dfrac{3 + \sqrt{-55}}{2} \\ \text{or } \dfrac{3 - \sqrt{-55}}{2} \end{matrix}\right\} \left.\begin{matrix} \text{or } \dfrac{3 - \sqrt{-55}}{2} \\ \text{or } \dfrac{3 + \sqrt{-55}}{2} \end{matrix}\right\}$$

EXERCISE 107

Solve:

1. $\begin{cases} x^2 + y^2 = 80 \\ x + y = 12. \end{cases}$

2. $\begin{cases} x^4 + y^4 = 97 \\ x + y = 5. \end{cases}$

3. $\begin{cases} x^3 + y^3 = 35 \\ x + y = 5. \end{cases}$

4. $\begin{cases} x^3 - y^3 = 56 \\ x - y = 2. \end{cases}$

5. $\begin{cases} x^3 + y^3 = 133 \\ x^2 - xy + y^2 = 19. \end{cases}$

6. $\begin{cases} x^3 - y^3 = 189 \\ x^2 + xy + y^2 = 63. \end{cases}$

7. $\begin{cases} x^2 + xy = 6 \\ xy + y^2 = 3. \end{cases}$

8. $\begin{cases} x^2 - xy = 3 \\ xy - y^2 = 2. \end{cases}$

9. $\begin{cases} x^5 + y^5 = 275 \\ x + y = 5. \end{cases}$

10. $\begin{cases} x^5 - y^5 = 242 \\ x - y = 2. \end{cases}$

11. $\begin{cases} x^2 + xy + y^2 = 37 \\ x^4 + x^2y^2 + y^4 = 481. \end{cases}$

12. $\begin{cases} x^2 - xy + y^2 = 19 \\ x^4 + x^2y^2 + y^4 = 729. \end{cases}$

IV. Special Methods of Solution.

148. As many simultaneous quadratics differ somewhat from the type forms enumerated, the student must exercise his ingenuity in devising the necessary modifications of the foregoing methods.

When one of a set of homogeneous equations is truly homogeneous, it may sometimes be factored, yielding two simple equations.

1. Solve $\begin{cases} 5x^2 - 7xy + 3y^2 = 4 & (1) \\ 3x^2 - 7xy + 4y^2 = 0 & (2) \end{cases}$

Factor ②, $(3x - 4y)(x - y) = 0$

Hence $3x - 4y = 0$ **and** $x - y = 0$

$$3x = 4y \qquad x = y$$

$$x = \frac{4y}{3}$$

Substitute these values of x **in** ①,

when $x = \frac{4y}{3}$, $\quad \frac{80y^2}{9} - \frac{28y^2}{3} + 3y^2 = 4.$

Clear of fractions, $80y^2 - 84y^2 + 27y^2 = 36$

$$23y^2 = 36$$

$$y^2 = \frac{36}{23}$$

$$y = \pm\tfrac{6}{23}\sqrt{23}$$

$$x = \tfrac{4}{3}y = \pm\tfrac{8}{23}\sqrt{23}$$

When $x = y$, $5y^2 - 7y^2 + 3y^2 = 4$

$$y^2 = 4$$

$$y = \pm 2$$

$$x = y \therefore x = \pm 2$$

When equations are symmetrical except as to algebraic signs, the method of symmetrical equations may be employed, that is, find the values of $x + y$ and $x - y$.

2. Solve $\begin{cases} x^3 - y^3 = 7xy & (1) \\ x - y = 2 & (2) \end{cases}$

Divide ① by ②, $x^2 + xy + y^2 = \frac{7}{2}xy$ ③

Square ②, $x^2 - 2xy + y^2 = 4$ ④

Subtract ④ from ③, $3xy = \frac{7}{2}xy - 4$

Transpose, $3xy - \frac{7}{2}xy = -4$

$$\frac{1}{2}xy = 4$$

$$xy = 8$$

Add $4xy$ to ④,

$$x^2 + 2xy + y^2 = 36$$

Extract square root, $x + y = \pm 6$ ⑤

Combine ⑤ and ②, $x - y = 2$ ②

By addition, $2x = 8 \text{ or } -4$

$$x = 4 \text{ or } -2$$

By subtraction, $2y = 4 \text{ or } -8$

$$y = 2 \text{ or } -4$$

3. Solve $\begin{cases} x^2 + y^2 = x^2y^2 + 1 & (1) \\ x + y = 2xy - 1 & (2) \end{cases}$

Square ②, $x^2 + y^2 + 2xy = 4x^2y^2 - 4xy + 1$ ③

Subtract ① from ③, $x^2 + y^2 = x^2y^2 + 1$ ①

$$2xy = 3x^2y^2 - 4xy$$

Transpose, $3x^2y^2 - 6xy = 0$

$$x^2y^2 - 2xy = 0$$

Factor, $xy(xy - 2) = 0$

$$xy = 0 \qquad xy = 2$$

Substitute these values of xy in ②,

$$x + y = -1 \text{ or } 3$$

When $xy = 0$, equation ① becomes

$$x^2 + y^2 = 1$$

Subtract $2xy = 0$, $x^2 - 2xy + y^2 = 1$

Extract square root, $x - y = \pm 1$

and $x + y = -1$

By addition, $2x = 0 \text{ or } -2$

$$x = 0 \text{ or } -1$$

By subtraction, $2y = -2 \text{ or } 0$

$$y = -1 \text{ or } 0$$

When $xy = 2$, equation ① becomes

$$x^2 + y^2 = 5$$

Subtract $2xy = 4$, $$x^2 - 2xy + y^2 = 1$$

$$x - y = \pm 1$$

$$x + y = 3$$

By addition, $$2x = 4 \text{ or } 2$$

$$x = 2 \text{ or } 1$$

By subtraction, $$2y = 2 \text{ or } 4$$

$$y = 1 \text{ or } 2$$

4. Solve $$\begin{cases} \dfrac{1}{x^2} + \dfrac{1}{y^2} = 25 & (1) \\ \dfrac{1}{x} - \dfrac{1}{y} = -1 & (2) \end{cases}$$

Square ②, $$\frac{1}{x^2} - \frac{2}{xy} + \frac{1}{y^2} = 1 \quad ③$$

Subtract ③ from ①, $$\frac{2}{xy} = 24 \quad ④$$

Add ④ to ①, $$\frac{1}{x^2} + \frac{2}{xy} + \frac{1}{y^2} = 49$$

Extract square root, $$\frac{1}{x} + \frac{1}{y} = \pm 7$$

Combine with ②, $$\frac{1}{x} - \frac{1}{y} = -1$$

By addition, $$\frac{2}{x} = 6 \text{ or } -8$$

$$\frac{1}{x} = 3 \text{ or } -4$$

$$x = \tfrac{1}{3} \text{ or } -\tfrac{1}{4}$$

By subtraction, $$\frac{2}{y} = 8 \text{ or } -6$$

$$\frac{1}{y} = 4 \text{ or } -3$$

$$y = \tfrac{1}{4} \text{ or } -\tfrac{1}{3}$$

When equations are symmetrical or symmetrical except for algebraic signs, the following method is sometimes preferable.

5. Solve $$\begin{cases} x^4 + y^4 = 82 & (1) \\ x - y = 2 & (2) \end{cases}$$

Let $x = u + v$ and $y = u - v$.

Substitute in ①, $(u + v)^4 + (u - v)^4 = 82$

$$u^4 + 4u^3v + 6u^2v^2 + 4uv^3 + v^4 + u^4 - 4u^3v + 6u^2v^2 - 4uv^3 + v^4 = 82$$

Combining like terms, $2u^4 + 12u^2v^2 + 2v^4 = 82$

Divide by 2, $u^4 + 6u^2v^2 + v^4 = 41$ ③

Substitute the values of x and y in ②,

$$u + v - u + v = 2$$
$$2v = 2$$
$$v = 1$$

Substitute this value of v in ③,

$$u^4 + 6u^2 + 1 = 41$$
$$u^4 + 6u^2 - 40 = 0$$

Factor, $(u^2 + 10)(u^2 - 4) = 0$

$$u^2 + 10 = 0 \qquad u^2 - 4 = 0$$
$$u^2 = -10 \qquad u^2 = 4$$
$$u = \pm\sqrt{-10} \qquad u = \pm 2$$

since $x = u + v$, $x = \pm\sqrt{-10} + 1$ or 3 or -1

and $y = u - v$, $y = \pm\sqrt{-10} - 1$ or 1 or -3

EXERCISE 108

Solve by any method:

1. $\begin{cases} x + y = 4 \\ xy = 3. \end{cases}$

2. $\begin{cases} x - y = 4 \\ xy = 5 \end{cases}$

3. $\begin{cases} x + y = 16 \\ x^2 + y^2 = 130. \end{cases}$

4. $\begin{cases} x - y = 8 \\ x^2 + y^2 = 34. \end{cases}$

5. $\begin{cases} x + 3y = -3 \\ x^2 + y^2 = 13. \end{cases}$

6. $\begin{cases} 2x - y = 11 \\ x^2 - y^2 = -16. \end{cases}$

7. $\begin{cases} 4x - 5y = -23 \\ 5x^2 - 4y^2 = -16. \end{cases}$

8. $\begin{cases} x^2 + xy + y^2 = 13 \\ x + y = 3. \end{cases}$

9. $\begin{cases} x^2 - xy + y^2 = 19 \\ x - y = 1. \end{cases}$

10. $\begin{cases} x^2 + xy + y^2 = 13 \\ x - y = 1. \end{cases}$

11. $\begin{cases} x^2 - 4xy + 3y^2 = 0 \\ 3x + 2y = 10. \end{cases}$

12. $\begin{cases} \frac{1}{x} + \frac{1}{y} = 5 \\ \frac{1}{x^2} + \frac{1}{y^2} = 13. \end{cases}$

13. $\begin{cases} x^2 - y^2 = 21 \\ x + y = -3. \end{cases}$

14. $\begin{cases} 2x^2 + 3y = 24 \\ 3x - 2y = -13. \end{cases}$

15. $\begin{cases} \frac{1}{x^2} - \frac{1}{y^2} = -\frac{3}{4} \\ \frac{1}{y} - \frac{1}{x} = \frac{1}{2}. \end{cases}$

16. $\begin{cases} 2x + 3y = -\frac{2xy}{5} \\ xy = -10. \end{cases}$

17. $\begin{cases} x^3 + y^3 = 56 \\ x + y = 2. \end{cases}$

18. $\begin{cases} x^3 - y^3 = 152 \\ x - y = 8. \end{cases}$

19. $\begin{cases} 2x^2 + xy - 3y^2 = 25 \\ 3x^2 - 2xy + y^2 = 57. \end{cases}$

20. $\begin{cases} 3x^2 + y^2 = 31 \\ 2xy + 3y^2 = 0. \end{cases}$

21. $\begin{cases} x^3 + y^3 = 26 \\ x^2 - xy + y^2 = 13. \end{cases}$

22. $\begin{cases} x^2 + xy = 8 \\ xy + y^2 = -4. \end{cases}$

23. $\begin{cases} x^2 - xy = 12 \\ xy - y^2 = 8. \end{cases}$

24. $\begin{cases} x^2 + 3xy = -5 \\ 3xy + 9y^2 = 6. \end{cases}$

25. $\begin{cases} x^2 + 5y^2 = 49 \\ 4xy - y^2 = -33. \end{cases}$

26. $\begin{cases} x^4 + y^4 = 17 \\ x + y = -1. \end{cases}$

27. $\begin{cases} x^5 - y^5 = 33 \\ x - y = 3. \end{cases}$

28. $\begin{cases} x^4 + y^4 = 97 \\ x - y = -5. \end{cases}$

29. $\begin{cases} x^2 + y^2 = xy + 19 \\ x + y = xy - 3. \end{cases}$

30. $\begin{cases} x^3 + y^3 = 9xy \\ x + y = 6. \end{cases}$

31. $\begin{cases} \dfrac{x+y}{x-y} + \dfrac{x-y}{x+y} = -8 \\ x^2 - y^2 = -3. \end{cases}$

32. $\begin{cases} x^2 + y^2 + x + y = 36 \\ xy = 10. \end{cases}$

33. $\begin{cases} x^2 - x^2y^2 + y^2 = 13 \\ x - 2xy + y = 8. \end{cases}$

34. $\begin{cases} x^4 + y^4 = 97 \\ x^2 + y^2 = 5xy + 43. \end{cases}$

35. $\begin{cases} x^2 - y^2 = a^2 \\ xy = c^2. \end{cases}$

36. $\begin{cases} x^3 + y^3 = m^3 \\ x + y = m. \end{cases}$

37. $\begin{cases} x^2 + xy = 2a^2 + ab \\ xy + y^2 = 2a^2 - ab. \end{cases}$

38. $\begin{cases} x^3 + y^3 = a^3 + b^3 \\ x^2y + xy^2 = a^2b + ab^2. \end{cases}$

39. $\begin{cases} x^2 - xy = m^2 - mn \\ xy - y^2 = mn - n^2. \end{cases}$

40. $\begin{cases} x + y + 3\sqrt{x+y} = 40 \\ x - y + 2\sqrt{x-y} = 8. \end{cases}$

PROBLEMS INVOLVING SIMULTANEOUS QUADRATICS

EXERCISE 109

1. The sum of two numbers is 7 and their product 12. Find the numbers.

Let x = one number and y = the other number.

Then, $x + y = 7$
And $xy = 12$

Solving by the method of symmetrical equations,

$$\begin{array}{rcl} x^2 + 2xy + y^2 &=& 49 \\ 4xy &=& 48 \\ \hline x^2 - 2xy + y^2 &=& 1 \\ x - y &=& \pm 1 \\ x + y &=& 7 \\ \hline 2x &=& 8 \text{ or } 6 \\ x &=& 4 \text{ or } 3 \\ 2y &=& 6 \text{ or } 8 \\ y &=& 3 \text{ or } 4 \end{array}$$

Hence, the numbers are 3 and 4.

2. The sum of two numbers is 17 and their product is 72. Find the numbers.

3. Find two numbers whose difference is 6 and whose product is 27.

4. Find two numbers whose sum is 9 and the difference of whose squares is 9.

5. The sum of the cubes of two numbers is 72 and their sum is 12. Find the numbers.

6. Find the dimensions of a rectangle whose perimeter is 44 ft. and whose area is 120 square feet.

7. The area of a rectangle is 1200 square feet and the length of the diagonal is 50 feet. Find the length and width.

8. Find the dimensions of a rectangular garden of half an acre, which requires 36 rods of fence.

9. A rectangular field having an area of 120 square rods is surrounded by a road 1 rod wide. The area of the road is 48 square rods. Find the dimensions of the field.

10. Two square gardens have together an area of 7400 square feet. A rectangular garden, whose dimensions are equal to the sides of the two squares, requires 240 feet of fence. Find the sides of the two squares.

11. A number is composed of two digits. The sum of the squares of the two digits is 73, and if 18 is added to the

number, the order of the digits will be reversed. Find the number.

12. The sum of the numerator and denominator of a fraction is 7. If the numerator is increased by 1 and the denominator diminished by 1, the resulting fraction will be the reciprocal of the original fraction. Find the fraction.

13. A tailor bought cloth for $400. Had he paid $1 more per yard, he would have received 20 yards less. Find how many yards he bought and the price per yard.

14. Two men working together can mow a field in 6 days. One man working alone can mow the field in 5 days less time than the other. In how many days can each alone do the work?

15. The front wheel of a carriage makes 48 revolutions more than the hind wheel in going one mile. If the circumference of each wheel were increased by 1 ft. the front wheel would make only 40 revolutions more than the hind wheel in going one mile. Find the circumference of each wheel.

16. A sum of money at interest amounted to $2120 at the end of one year. Had the principal been $400 more and the rate 1 percent less, the interest would have been the same. Find the principal and the rate of interest.

17. A man has a distance of 20 miles to go. Part of the distance he walks, and part he rides in a borrowed motor car which runs 4 times as fast as he can walk, the trip taking him 3 hours and 10 minutes. The return trip, being down hill, he makes in 2 hours and 30 minutes, the motor running 2 miles an hour faster and his rate of walking increased one mile per hour. Find the distance he walks and his rate of walking and riding.

18. Two athletes run a 100-yard race. In the first heat, A is beaten by B by $1\frac{7}{18}$ seconds. In the second heat, each increases his speed by one yard per second and A is beaten by $1\frac{1}{9}$ seconds. Find the speed at which each ran.

GRAPHICAL SOLUTION OF SIMULTANEOUS QUADRATICS

149. As in the graphical solution of simultaneous simple equations, the values of x and y that will satisfy both equations are the coördinates of the points that lie in both graphs, that is, the coördinates of the points of intersection. If the graphs do not intersect, the equations have no real roots.

1. Solve $\begin{cases} x^2 + y^2 = 25 & (1) \\ 2x + 3y = 18 & (2) \end{cases}$

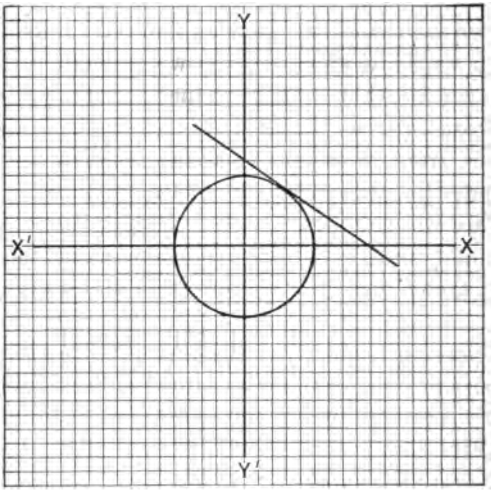


Fig. I.

In (1) $y^2 = 25 - x^2$

$$y = \pm\sqrt{25 - x^2}$$

when $x = 0,\ y = \pm 5$

$x = \pm 1,\ y = \pm\sqrt{24} = \pm 4.90$

$x = \pm 2,\ y = \pm\sqrt{21} = \pm 4.58$

$x = \pm 3,\ y = \pm\sqrt{16} = \pm 4$

$x = \pm 4,\ y = \pm\sqrt{9} = \pm 3$

$x = \pm 5,\ y = 0$

If x were greater than 5 or less than -5, y would be imaginary. Fig. 1 shows that the graph of (1) is a circle.

In (2),

When $x = 0,\ y = 6$

When $y = 0,\ x = 9$

The graph of (2) is a straight line, touching the circle at one point only, when $x = 3$ and $y = 4$. Hence these two values are the only real roots of these equations.

2. Solve $\begin{cases} 5x^2 - 4y^2 = 80 & (1) \\ x^2 + 6y^2 = 64 & (2) \end{cases}$

In ①, if $x = 0$ y is imaginary, since $y = \pm\sqrt{\frac{5x^2 - 80}{4}}$.

when x is less than ± 4, y is imaginary.

$$x = \pm 4, \quad y = 0$$

$$x = \pm 5, \quad y = \pm\sqrt{\frac{125 - 80}{4}} = \pm 3.3+$$

$$x = \pm 6, \quad y = \pm\sqrt{\frac{180 - 80}{4}} = \pm 5$$

$$x = \pm 7, \quad y = \pm\sqrt{\frac{245 - 80}{4}} = \pm 6.4+$$

$$x = \pm 8, \quad y = \pm\sqrt{\frac{320 - 80}{4}} = \pm 7.7+$$

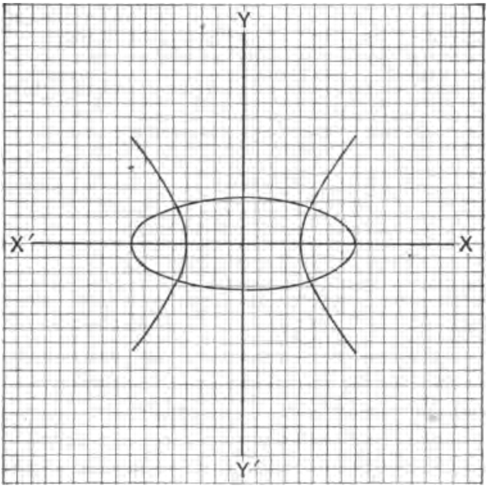


Fig. II.

For larger values of x, y will become larger, and the graph of ① is an hyperbola. Fig. 2.

In ②, if $x = 0$, $y = \pm\sqrt{\frac{64}{6}} = \pm 3.2+$

if $y = 0$, $x = \pm\sqrt{64} = \pm 8$

Since $y = \pm\sqrt{\dfrac{64 - x^2}{6}}$, any value of x greater than $+8$ or less than -8, will make y imaginary; hence the graph must lie between $x = +8$ and $x = -8$.

If $x = \pm 1$, $y = \pm\sqrt{\frac{63}{6}} = \pm 3.2+$
$x = \pm 2$, $y = \pm\sqrt{\frac{60}{6}} = \pm 3.1+$
$x = \pm 3$, $y = \pm\sqrt{\frac{55}{6}} = \pm 3.0+$
$x = \pm 4$, $y = \pm\sqrt{\frac{48}{6}} = \pm 2.8+$
$x = \pm 5$, $y = \pm\sqrt{\frac{39}{6}} = \pm 2.5+$
$x = \pm 6$, $y = \pm\sqrt{\frac{28}{6}} = \pm 2.1+$
$x = \pm 7$, $y = \pm\sqrt{\frac{15}{6}} = \pm 1.5+$

The graph of (2) is an ellipse.

There are 4 points of intersection, hence the roots are $x = 4.5$ approx., $y = \pm 2.5$ approx., and $x = -4.5$, $y = \pm 2.5$.

3. Solve $\begin{cases} x^2 + y^2 = 36 & (1) \\ 3y^2 - 10x = 0 & (2) \end{cases}$

The graph of (1) is a circle found as in Example 1. Fig. 3.

In ②, $x = \dfrac{3y^2}{10}$. In this case it is more convenient to assume values of y and find the corresponding values of x.

when $y = 0$ $x = 0$
$y = \pm 1$ $x = .3$
$y = \pm 2$ $x = 1.2$
$y = \pm 3$ $x = 2.7$
$y = \pm 4$ $x = 4.8$
$y = \pm 5$ $x = 7.5$
$y = \pm 6$ $x = 10.8$

For larger values of y, we get larger values of x which bring the graph beyond the circle.

The graph of (2) is a parabola.

In this case, there are only two points of intersection, giving two values of y and one value of x.

When $x = 4.5$ approx., $y = \pm 3.8$ approx.

The student will note that every quadratic equation will yield one of the curves known as the conic sections, that is a circle, parabola, ellipse or hyperbola.

The position of these curves, with reference to the axes of x and y, may be shifted, depending upon the form of the equations.

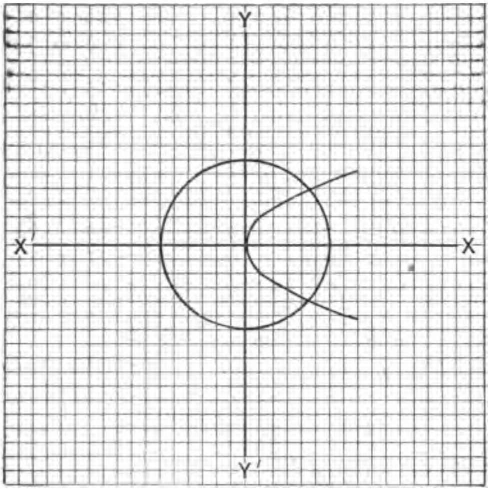


FIG. III.

EXERCISE 110

Solve by graphs:

1. $\begin{cases} x^2 + y^2 = 25 \\ x + y = 3 \end{cases}$

2. $\begin{cases} x - y = 6 \\ xy = 8 \end{cases}$

3. $\begin{cases} x^2 - y^2 = 9 \\ x - y = 1 \end{cases}$

4. $\begin{cases} x^2 + 3y = 12 \\ 3x - y = 3 \end{cases}$

5. $\begin{cases} x^2 + y^2 + 9x = -14 \\ y^2 - 4x = 16 \end{cases}$

6. $\begin{cases} x^2 + 4y^2 = 17 \\ xy = 2 \end{cases}$

7. $\begin{cases} x^2 + y^2 = 36 \\ xy = 14 \end{cases}$

8. $\begin{cases} x^2 + y^2 + x + y = 6 \\ xy + x + y = 2 \end{cases}$

9. $\begin{cases} x^2 + 4y = 4 \\ x^2 + y^2 = 16 \end{cases}$

10. $\begin{cases} x^2 - xy + y^2 = 31 \\ x^2 + y^2 = 61 \end{cases}$

REVIEW EXERCISE XVIII

Solve:

1. $\begin{cases} x - 9 = y \\ xy + 8 = 0 \end{cases}$

2. $\begin{cases} x + y = a \\ x^2 + y^2 = b \end{cases}$

3. $\begin{cases} x^2 + xy + y = 137 \\ y^2 + xy + x = 205 \end{cases}$

4. $\begin{cases} \dfrac{x^2}{y} + \dfrac{y^2}{x} = \dfrac{7}{2} \\ x + y = 1 \end{cases}$

5. $\begin{cases} x^2 + xy + y^2 = 37 \\ x^4 + x^2y^2 + y^4 = 481 \end{cases}$

6. $\begin{cases} xy - (x - y) = 1 \\ x^2y^2 + (x - y)^2 = 13 \end{cases}$

7. $\begin{cases} \dfrac{1}{x^2} + \dfrac{1}{y^2} = 65 \\ \dfrac{1}{x} - \dfrac{1}{y} = 11 \end{cases}$

8. $\begin{cases} 3x^2 + xy + y^2 = 47 \\ 4x^2 - 3xy - y^2 = -39 \end{cases}$

9. $\begin{cases} \sqrt{x} + \sqrt{y} = 5 \\ xy = 36 \end{cases}$

10. $\begin{cases} x^2 + xy + y^2 = 84 \\ x + \sqrt{xy} + y = 6 \end{cases}$

11. $\begin{cases} x^3 - y^3 = 152 \\ x = 8 + y \end{cases}$

12. $\begin{cases} 2x^3 + 2y^3 = 9xy \\ x + y = 3 \end{cases}$

13. $\begin{cases} 3x^2 + xy + 3y^2 = 8\frac{1}{4} \\ 8x^2 - 3xy + 8y^2 = 17\frac{3}{4} \end{cases}$

14. $\begin{cases} xy + x + y = 7 \\ x^2y + xy^2 = 12 \end{cases}$

15. $\begin{cases} x^2 - xy = a^2 + b^2 \\ xy - y^2 = 2ab \end{cases}$

16. $\begin{cases} x^2 + xy = 40 \\ 27 + 2y^2 = 3xy \end{cases}$

17. $\begin{cases} x^2 + 2xy = -21 \\ y^2 + 2xy = -5 \end{cases}$

18. $\begin{cases} x^2 - xy - 2y^2 = 0 \\ x^2 + y = 5 \end{cases}$

19. $\begin{cases} x + y = 13 + \sqrt{xy} \\ x^2 + y^2 = 273 - xy \end{cases}$

20. $\begin{cases} x^2 - xy - 2y^2 = 0 \\ 2x^2 - 3x + y = 3 \end{cases}$

21. $\begin{cases} x^2 + xy + 2y^2 = 74 \\ 2x^2 + 2xy + y^2 = 73 \end{cases}$

22. $\begin{cases} x^2 + y^2 - 5x + 5y = 30 \\ xy = 8 \end{cases}$

23. $\begin{cases} x^2 - 2xy - 4y^2 = -41 \\ x^2 - 5xy + 8y^2 = 58 \end{cases}$

24. $\begin{cases} xy(x + y) = 30 \\ x^3 + y^3 = 35 \end{cases}$

25. $\begin{cases} 3x^2 + 3y^2 - 5xy = 15 \\ 3x + 3y = 4xy \end{cases}$

Solve graphically:

26. $\begin{cases} x^2 + 2y^2 = 99 \\ x - 2y = 3 \end{cases}$

27. $\begin{cases} x^2 + xy + y^2 = 19 \\ x^3 - y^3 = 19 \end{cases}$

28. $\begin{cases} x^2 - 4y = -7 \\ 2x + 3y = 4 \end{cases}$

29. $\begin{cases} y^2 - 4x = 0 \\ 3y^2 + 2x^2 = 14 \end{cases}$

30. $\begin{cases} x^2 - 2x - y = 3 \\ 2x + y = 1 \end{cases}$

31. A number is formed of two digits. The sum of the squares of the digits is 58. If 12 times the units' digit is subtracted from the number, the order of the digits will be reversed. Find the number.

32. A rectangular garden is surrounded by a walk 7 feet wide. The area of the garden is 15,000 square feet, and of the walk 3696 square feet. Find the dimensions of the garden.

33. The sum of two numbers is 9, and the sum of their cubes is 243. Find the numbers.

34. The difference between the areas of two squares is 231 square inches. The difference between their perimeters is 28 inches. Find their sides.

35. The hypotenuse of a right triangle is 20 inches and the area of the triangle is 96 square inches. Determine the legs.

36. If the difference of two numbers multiplied by the greater is 160, and multiplied by the less, is 96, find the numbers.

CHAPTER XIX

LOGARITHMS

150. The **logarithm** of any number is the *exponent* of the power to which a fixed number must be raised to equal the given number. The fixed number is called the **base** of the system of logarithms.

The **Briggs** or **Common** system of logarithms uses the number 10 as its base. Hence, the common logarithm of any number is the exponent of the power to which 10 is raised to equal the given number. Thus since $100 = 10^2$, the logarithm of 100 is 2, which is written, $\log 100 = 2$. In this book, the common system of logarithms only is used.

$$\begin{array}{llll}
\text{Since} & 10^0 = 1 & \therefore & \log 1 = 0 \\
 & 10^1 = 10 & \therefore & \log 10 = 1 \\
 & 10^2 = 100 & \therefore & \log 100 = 2 \\
 & 10^3 = 1000 & \therefore & \log 1000 = 3 \\
 & 10^4 = 10000 & \therefore & \log 10000 = 4
\end{array}$$

and so on.

$$\begin{array}{llll}
\text{Also, since} & 10^{-1} = \frac{1}{10} & \therefore & \log .1 = -1 \\
 & 10^{-2} = \frac{1}{100} & \therefore & \log .01 = -2 \\
 & 10^{-3} = \frac{1}{1000} & \therefore & \log .001 = -3 \\
 & 10^{-4} = \frac{1}{10000} & \therefore & \log .0001 = -4
\end{array}$$

and so on.

Hence, the logarithms of numbers greater than 1 are positive, and the logarithms of numbers less than 1 are negative. Also, since most numbers are not an integral power of 10, the logarithms of such numbers are not integers, but mixed numbers, consisting of an integral part and a fraction. This fractional part of the logarithm is usually expressed as a decimal.

JOHN NAPIER
(From an Engraving After an Original Edinburgh Painting)

Born near Edinburgh, 1550. Died, 1617.
Inventor of logarithms.

His *Mirifici logarithmorum canonis descriptio* (1614), the first important work on mathematics produced in Great Britain, inspired Briggs to develop the common system of logarithms with the decimal base.

Napier was a student of theology and astrology, engaging in many controversies.

The great object of his mathematical studies was to simplify and systematize arithmetic, algebra and trigonometry.

Napier succeeded in constructing a system of logarithms before exponents were used.

The integral part of the logarithm is called the **characteristic**, and the decimal part is called the **mantissa** of the logarithm.

Since the log 1 = 0 and log 10 = 1, the logarithms of the numbers between 1 and 10 must lie between 0 and 1, that is, the logarithm of any number, having one figure in the whole number, is some decimal fraction.

Similarly, since log 10 = 1 and log 100 = 2, the logarithms of numbers between 10 and 100 must lie between 1 and 2, that is, the logarithm of any number, having two figures in the whole number, is 1 plus some decimal. In the same way it may be shown, that the logarithm of any number having three figures in the whole number is 2 plus some decimal.

Hence, the following rule:

The characteristic of the logarithm of any number greater than 1 *is positive and* 1 *less than the number of figures in the integral part of the number.*

Note that it is the integral part of the logarithm which is called the characteristic. An ordinary number has no characteristic. Thus, if the log 32.56 = 1.51268, 1 is the characteristic of the logarithm, while 32 is simply the integral part of the number 32.56.

Since the log 1 = 0 and log .1 = −1, the logarithms of numbers between 1 and .1 lie between 0 and −1, that is, the logarithm of any decimal fraction, whose first significant figure is in the first place of decimals, is a negative decimal.

Similarly, since the log .1 = −1 and log .01 = −2, the logarithms of numbers between .1 and .01 lie between −1 and −2, that is, the logarithm of any decimal fraction whose first significant figure is in the second place of decimals is −1 plus some negative decimal.

Instead of writing these logarithms entirely as negative numbers, it is found more convenient to express these logarithms in the form of a negative characteristic plus a positive decimal fraction; that is, the mantissa is always positive.

Thus, the log .03256 = -1.48732 but is expressed as $-2 + .51268$ which is equal to -1.48732.

Therefore, the logarithm of any number between 1 and .1 has the form $-1 +$ some positive decimal, the logarithm of any number between .1 and .01 has the form $-2 +$ some positive decimal and so on.

Hence, the following rule:

The characteristic of the logarithm of any number less than 1 is negative and equal to the order of the place of the first significant figure.

Thus, the characteristic of the logarithm of .00456 is -3, since the first significant figure, 4, is in the third place of decimals.

For convenience, a negative characteristic is usually written in two parts by adding and subtracting 10 or some multiple of 10.

Thus, the logarithm $-1 + .52368$ is written $9.52368 - 10$, $9 - 10$ being equal to -1.

Since all logarithms are exponents, they can be used only in arithmetical operations that can be performed by means of exponents, as explained in the chapter on Theory of Exponents. Logarithms may therefore be used in any computation involving multiplication, division, involution or evolution.

If the number $32.56 = 10^{1.51268}$, log $32.56 = 1.51268$. Then $32.56 \times 10 = 10.^{1.51268} \times 10^{1} = 10^{2.51268}$ that is, $325.6 = 10^{2.51268}$ $\therefore$ log $325.6 = 2.51268$. Also, $32.56 \div 1000 = 10^{1.51268} \div 10^{3} = 10^{-2+.51268}$, that is, $.03256 = 10^{-2+.51268}$ $\therefore$ log $.03256 = -2 + .51268$.

This shows that a multiplication or division by any power of 10 does not change the mantissa of the logarithm. But a multiplication or division of any number by a power of 10 merely shifts the position of the decimal point, to the right in multiplication, to the left in division.

Hence, the mantissa is not changed by the shifting of the decimal point, that is, *the value of the mantissa depends only on the combination of figures in the number*, and will remain constant, whether the combination of figures represents a whole number, a mixed number, or a decimal fraction.

Since the characteristic of the logarithm of any number can be determined by inspection according to the rules given, it is only necessary to find the mantissa.

The mantissas for various combinations of figures have been computed and may be found in the table of logarithms, page 361.

USE OF THE TABLE OF LOGARITHMS

151. This table of logarithms gives the mantissa, to five decimals, for any combination of four figures.

The first three figures are given in the column at the left of the page, headed **No**, and the fourth figure is placed at the top of the ten columns, 0 to 9.

1. Find log 3743.

The characteristic of the log 3743 is 3, one less than the number of figures in the whole number.

On page 366 of the table, we find the first three figures 374 in column **No**, and in line with 374, in column headed 3, we find the mantissa 57322, hence log 3743 = 3.57322.

2. Find log 57.64.

The characteristic of log 57.64 is 1.

The mantissa is found on page 370, in line with 576, column headed 4, 76072.

Hence, log 57.64 = 1.76072.

3. Find log. .001236.

The characteristic of log .001236 is −3, written 7 − 10.

The mantissa for the combination 1236 is found on page 361, in line 123, column headed 6, 09202.

Hence log .001236 = 7.09202 − 10.

4. Find log 4.7757.

The characteristic of log 4.7757 is 0.

As our combination consists of five significant figures, and only four are given in the table, a process, called **interpolation**, is necessary.

On page 368, we find the mantissa for 4775, the first four figures of the combination, to be .67897.

The mantissa for 4776 is .67906. Hence, for an additional unit in the combination of four figures, the mantissa is increased by

.00009. This difference between any two successive mantissas is called the **tabular difference,** and is given in the column headed D at the right of the page. In some cases, the difference between two successive mantissas may vary by a unit from that given in the column of differences, but the use of the tabular difference will not introduce any serious error in the result of the computation.

In any combination of figures, the actual value of a figure depends upon its position in the combination, that is, the value of a unit in any position is ten times the value it would have if moved one place to the right. Hence, the fifth figure, 7, of the combination 47757 is only seven-tenths of a unit in the fourth place of the combination. If we assume, that, between two consecutive mantissas, the change is directly proportional to the increase in the number, then, to get the mantissa for 47757, we must add seven-tenths of the tabular difference to the mantissa for 4775.

In interpolation, it is convenient to treat mantissas as whole numbers.

Thus, the mantissa for 4775 is 67897.

The tabular difference is 9.

.7 of the tabular difference, $9 \times .7 = 6.3$.

Discarding the decimal .3, as it is less than .5, we add the difference 6 to the mantissa 67897.

Hence, $\log 4.7757 = 0.67903$.

The student should note that all logarithms are approximations only, in our table, correct to five decimals. When greater accuracy is required, tables, giving mantissas to a greater number of decimal places, must be used.

5. Find log 0.0002457689.

The characteristic of log 0.0002457689 is -4 or $6 - 10$.

The mantissa for 2457 is 39041.

The tabular difference is 18.

As the last three figures of our combination of figures are .689 of a unit in the fourth place, the tabular difference, 18, is multiplied by .689. ∴ add 12.

Hence $\log 0.0002457689 = 6.39053 - 10$.

EXERCISE 111

Find logarithms of:

1. 5512.
2. 12.40.
3. .6892.
4. 1743.5.
5. .047316.
6. 78000.
7. 3.1416.
8. .7854.
9. 313476.
10. 678.5478.

TO FIND THE NUMBER CORRESPONDING TO A GIVEN LOGARITHM, CALLED THE ANTILOGARITHM

152. 1. Find antilog 2.23930.

First find the combination of figures corresponding to the mantissa 23930.

This mantissa is found on page 362, the number corresponding is 1735.

The characteristic, 2, shows that the number has three figures in the whole number.

Hence antilog 2.23930 = 173.5.

2. Find antilog 7.44154 − 10.

On page 364, we find the mantissa 44154 and the number corresponding, 2764.

The characteristic 7 − 10 or −3 shows that the number is a decimal and that the first significant figure is in the third place of decimals.

Hence antilog 7.44154 − 10 is .002764.

3. Find antilog 3.67435.

The mantissa 67435 does not occur in the table. This indicates that the number corresponding is a group of more than four figures.

The *nearest less* mantissa on page 368, is 67431 which differs from the given mantissa by 4.

The tabular difference is given as 9, which would be the difference to be added for one unit in the fourth place. As our difference is only 4, it follows that the number corresponding to the given mantissa is only $\frac{4}{9}$ of a unit greater than the combination 4724, corresponding to the mantissa 67431.

The fraction $\frac{4}{9}$ is changed to a decimal by dividing the numerator by the denominator; $\frac{4}{9} = .444 +$. Hence the number corresponding to the given mantissa is .444 + of a unit greater than the combination 4724. The fraction .444 + is added by simply annexing it to the first four figures of the combination.

That is, the mantissa 67435 corresponds to the combination 4724444 +.

Hence antilog 3.67435 = 4724.444.

To find the additional figures of any combination of figures corresponding to a given mantissa, divide the difference between the given mantissa and the nearest less mantissa in the table by the tabular difference. Any number of additional figures may be found by carrying out the division to any required number of decimal places.

EXERCISE 112

Find the antilogarithms of:

1. 1.61109.
2. 0.69399.
3. 7.32160 − 10.
4. 3.45380.
5. 5.93584.
6. 9.57342 − 10.
7. 6.31485 − 10.
8. 5.43782.
9. 0.67345.
10. 8.22313 − 10.

COMPUTATIONS BY LOGARITHMS

153. Since all logarithms are exponents, the theory of exponents is applied in computations by logarithms.

1. Multiply, 304.67×0.004678.

Let $x = 304.67 \times 0.004678$.

Then $\log x = \log(304.67 \times 0.004678) = \log 304.67 + \log 0.004678$ (In multiplication add exponents.)

$$\begin{aligned} \log 304.67 &= 2.48383 \\ \log 0.004678 &= \underline{7.67006 - 10} \\ & \quad 10.15389 - 10 \\ \therefore \log x &= 0.15389 \end{aligned}$$

Nearest less mant. = $\underline{15381}$ and number corresponding = 1425

$$\begin{aligned} \text{Dif.} &= 8 \\ \text{Tab. Dif.} &= 31 \\ 8 \div 31 &= .258 \end{aligned}$$

Hence, number corresponding to 15389 is 1425258 and characteristic being 0, $x = 1.425258$.

Note.—The number of decimals to which the result should be carried out is usually determined by the number of decimals in the given numbers. In this problem, the given numbers contained six decimal places, hence the result should be accurate to six decimals.

2. Divide 34 by .004756.

Let $x = 34 \div .004756$.

Then $\log x = \log 34 - \log .004756$.

$$\begin{aligned} \log 34 &= 11.53148 - 10 \text{ (Mantissa of 34 equals mantissa for 3400)} \\ \log .004756 &= \underline{7.67724 - 10} \\ \log x &= 3.85424 \end{aligned}$$

To be able to subtract the logarithm 7.67724 − 10 from the logarithm 1.53148, add 10 − 10 which equals 0 and will not change the value of log 34.

The nearest less mantissa to 85424 is 85418 corresponding to the figures 7148.

The difference of the mantissas is 6 and the tabular difference is given as 6. The actual difference between the two successive mantissas in the table is 7. Ordinarily, the difference given in the column of differences may be used as the error would be negligible. In this case, as the characteristic is 3, the number has four figures to the left of the decimal point, and it is better to use the actual difference 7.

$$6 \div 7 = .857143$$

Hence the mantissa 85424 corresponds to the combination 7148857143, and $x = 7148.857143$.

3. Find the cube of 16.235.

Let $x = (16.235)^3$

Then $\log x = 3 \log 16.235$.

$$\begin{array}{rrl} \log 16.235 = & 1.21046 & \\ \text{Multiplied by } 3, & 3 & \\ \hline \log x = & 3.63138 & \\ \text{Nearest mantissa} = & 63134 & \text{corresponds to } 4279 \\ \hline \text{Dif.} = & 4 & \\ \text{Tab. Dif.} = & 10 & \\ \text{And} \quad 4 \div 10 = & .4 & \end{array}$$

Hence 63138 corresponds to 42794.

And $x = 4279.4$.

4. Find the 7th root of .08456.

Let $x = \sqrt[7]{.08456}$

Then $\log x = \frac{1}{7} \log .08456$

$\log .08456 = 8.92716 - 10$

To make the negative part of the characteristic divisible by 7, add 60 − 60.

$$\begin{array}{rll} 7\overline{)68.92716 - 70} & & \\ \log x = & 9.84674 - 10 & \\ \text{Nearest less} = & 84671 & \text{corresponds to } 7026 \\ \hline \text{Dif.} = & 3 & \\ \text{Tab. Dif.} = & 6 & 3 \div 6 = .5 \\ \therefore\ x = & .70265 & \end{array}$$

EXERCISE 113

Find, by logarithms, the value of:

1. 65.02×97.98.
2. 7.298×0.04754.
3. $2.057 \div 0.4963$.
4. $34.29 \div 37.69$.
5. 5.032^3.
6. 0.7583^4.
7. $\sqrt[5]{9.068}$.
8. $\sqrt[6]{.008712}$.

9. $35.672 \times .0087932$.
10. $58 \div 2.6945$.
11. 3.1416^3.
12. 0.0058936^2.
13. $\sqrt[4]{.054}$.
14. $\sqrt[8]{.134678}$.
15. $\sqrt[3]{.9}$.
16. $8 \times .34 \times 576.38$.

COLOGARITHMS

154. In problems of multiplication and division, involving more than two numbers, it is convenient to change divisions to multiplications, and then finding the logarithm of the result by a single addition. It is evident that any division may be changed to a multiplication by using the reciprocal of the divisor.

Thus $3 \div 4 = 3 \times \frac{1}{4}$

And $\log (3 \div 4) = \log (3 \times \frac{1}{4}) = \log 3 + \log \frac{1}{4}$.

The logarithm of the reciprocal of a number is called the **cologarithm** of the number. Thus the colog 4 is equal to $\log \frac{1}{4}$.

But $\log \frac{1}{4} = \log 1 - \log 4$ and as $\log 1 = 0$, it follows that the cologarithm of any number may be found by subtracting the logarithm of the number from 0. This is most conveniently done by subtracting the logarithm of the number from $10 - 10$ or a multiple of $10 - 10$.

$$\text{Thus, colog } \tfrac{1}{4} = \begin{array}{rl} 10 & -10 \\ -\ 0.60206 & \\ \hline 9.39794 & -\ 10 \end{array}$$

A little practice will enable the student to find the cologarithm of a number almost as quickly as the logarithm.

EXERCISE 114

Find the cologarithm of:

1. 3467.
2. .5.
3. 1.674.
4. .005738.
5. 45.685.
6. 934678.
7. .000013.
8. 234.16.
9. 192,000,000.
10. 367.587.

EXTENDED OPERATIONS

155. 1. Find value of: $\sqrt[5]{\dfrac{67 \times 348.6 \times 53}{17^2 \times \sqrt[3]{789} \times \sqrt{4873}}}$

Let x = the required value,

Then, $\log x = \frac{1}{5}(\log 67 + \log 348.6 + \log 53 + 2 \text{ colog } 17 + \frac{1}{3} \text{ colog } 789 + \frac{1}{2} \text{ colog } 4873)$.

$$\begin{array}{rcl@{\quad}c@{\quad}l}
\log 67 & = & 1.82607 & = & 1.82607 \\
\log 348.6 & = & 2.54233 & = & 2.54233 \\
\log 53 & = & 1.72428 & = & 1.72428 \\
2 \text{ colog } 17 & = & 2(8.76955 - 10) & = & 7.53910 - 10 \\
\frac{1}{3} \text{ colog } 789 & = & \frac{1}{3}(7.10292 - 10) & = & 9.03431 - 10 \\
\frac{1}{2} \text{ colog } 4873 & = & \frac{1}{2}(6.31220 - 10) & = & \underline{8.15610 - 10} \\
 & & & & 30.82219 - 30
\end{array}$$

$$\begin{aligned}
5 \log x &= 0.82219 \\
\log x &= 0.16444 \\
\therefore x &= 1.4603
\end{aligned}$$

Note.—In operations performed by logarithms, algebraic signs will have no effect on the numerical value of the result, but the algebraic sign of the result must be determined by inspection.

EXERCISE 115

Find, by logarithms, the value of:

1. $(68.3 \times 0.46) \div 3.758$.
2. $6.743 \div (36.4 \times 5.76)$.
3. $(-0.0042 \times 674.358) \div (-42.3)$.
4. $357.7069 \div (7.94889 \times 28315)$.
5. $.0051988 \div (-1781 \times 0.00080005)$.
6. $\dfrac{84.9707 \times .343709 \times 8.67383}{750009 \times .03768593 \times 8.97476}$.
7. $\dfrac{48.668 \times 734.4378 \times 7.3845}{.83746 \times .573145 \times .0060788}$.
8. $\dfrac{.70383 \times .601978 \times .00653909}{761.25 \times 67.6785 \times 3947.58}$.
9. $\sqrt[4]{\dfrac{79.325 \times 0.4628 \times 80.644}{5 \times .5 \times .0005}}$.
10. $\sqrt[3]{.2 \times \frac{1}{2} \times .22 \times .0002}$

11. $\sqrt{\dfrac{14.163^2 \times 1774.3 \times .54778^3}{.093467 \times \sqrt{6784.5} \times \sqrt[3]{9.6745}}}$.

12. $\sqrt[5]{\dfrac{.00032 \times \sqrt{.00016} \times \sqrt[3]{.00027}}{1000 \times \sqrt{.4} \times \sqrt[3]{.8}}}$.

EXPONENTIAL EQUATIONS

156. Equations, in which the unknown quantity occurs as an exponent, are called **exponential equations.**

Such equations can usually be solved by means of logarithms.

1. Solve $(1.4)^x = 3.5$.

By logarithms, $\log (1.4)^x = \log 3.5$

$$\therefore x \log 1.4 = \log 3.5$$

and dividing by cœff. of x, $x = \dfrac{\log 3.5}{\log 1.4} = \dfrac{0.54407}{0.14613}$

This division may be performed by logarithms.

$$\begin{aligned} \log 0.54407 &= 9.73566 - 10 \\ -\log 0.14613 &= 9.16474 - 10 \\ \log x &= 0.57092 \\ \therefore\ x &= 3.72325 \end{aligned}$$

2. Solve, $.05^x = .32$.

$$x \log .05 = \log .32$$

$$x = \frac{\log .32}{\log .05} = \frac{9.50515 - 10}{8.69897 - 10}$$

To perform this division, the actual values of these numbers must be found.

$$x = \frac{9.50515 - 10}{8.69897 - 10} = \frac{-0.49485}{-1.30103}$$

Then by logarithms, $\log x = \log .49485 - \log 1.30103$

$$\begin{aligned} \log .49485 &= 9.69448 - 10 \\ -\log 1.30103 &= 0.11429 \\ \log x &= 9.58019 - 10 \\ x &= 0.3803545 \end{aligned}$$

3. Find the logarithm of $\frac{1}{27}$ to the base 3.

If the base of the system of logarithms is something besides 10, it is usually indicated by a small figure written below and to the right of the abbreviation *log*.

The problem may then be written, find $\log_3 \frac{1}{27}$.

This leads to the solution of an exponential equation, as it is required to find the exponent of the power to which 3 must be raised to equal $\frac{1}{27}$.

Hence $$3^x = \tfrac{1}{27}.$$

This equation may be solved in the same way as the preceding problems, but whenever the numbers on each side of the equation can be expressed as powers of the same number, the equation can be solved by inspection without the use of logarithms.

Thus $$3^x = \frac{1}{3^3} = 3^{-3}$$

Then by inspection, $$x = -3.$$

4. Solve $ab^x = cd^{x+2}$.

By logarithms, $\log a + x \log b = \log c + (x + 2) \log d$

Simplifying, $\log a + x \log b = \log c + x \log d + 2 \log d$

Transpose, $x \log b - x \log d = \log c + 2 \log d - \log a$

$x(\log b - \log d) = \log c + 2 \log d - \log a$

Divide by coefficient of x, $$x = \frac{\log c + 2 \log d - \log a}{\log b - \log d}$$

EXERCISE 116

Solve:

1. $3^x = 6$.
2. $5^{2x} = 20$.
3. $2.4^x = 4.8$.
4. $(.5)^{-x} = 3$.
5. $(.7)^{-3x} = 5$.
6. $\log_2 32$.
7. $\log_{\frac{1}{2}} 8$.
8. $\log_2 \frac{1}{64}$.
9. $\log_5 100$.
10. $\log_{.3} 3$.
11. $ab^x = c$.
12. $m^{x-1} = n$.
13. $ac^{1-x} = 2bd^{x-2}$.
14. $pq^{2x} = m^x n^{3x}$.
15. $ab^x = 5c^3$.

COMPOUND INTEREST

157. Let P = the principal at compound interest

r = the rate per cent

n = the number of years

A = the amount at the end of n years.

Then at the end of the first year, $A = P(1 + r)$

at the end of the 2d year, $A = P(1 + r)^2$

And at the end of the nth year, $A = P(1 + r)^n$

This formula may be used in computing the amount of any sum at compound interest.

1. Find the amount of \$200 for 8 years at 5 per cent compound interest.

Here $P = \$200$
$n = 8$
$r = 5\%$

$$\begin{aligned} \text{hence, } A &= 200(1.05)^8 \\ \text{and } \log A &= \log 200 + 8 \log 1.05 \\ \log 200 &= 2.30103 \\ 8 \log 1.05 &= \underline{0.16952} \\ \log A &= 2.47055 \\ \therefore A &= \$295.49 \end{aligned}$$

If the interest is compounded semi-annually, the rate is halved and the number of interest periods doubled; that is

$$A = P\left(1 + \frac{r}{2}\right)^{2n}.$$

2. Find the principal which will amount to \$1000 in 5 years at 6 per cent compound interest.

$A = \$1000$
$n = 5$ yrs.
$r = 6\%$
P is required

$$\begin{aligned} A &= P(1 + r)^n \\ 1000 &= P(1.06)^5 \\ P &= \frac{1000}{(1.06)^5} \\ \log P &= \log 1000 - 5 \log 1.06 \\ \log 1000 &= 3.00000 \\ 5 \log 1.06 &= \underline{0.12655} \\ \log P &= 2.87345 \\ \therefore P &= \$747.22 \end{aligned}$$

EXERCISE 117

1. Find the amount of \$150 for 7 years at 4 per cent compound interest.

2. Find the amount of \$500 for 12 years at 5 per cent compound interest.

3. Of \$300 for 9 years 9 months at 5.4 per cent compound interest.

4. Of \$600 for 8 years at 6 per cent compounded semi-annually.

5. Find the principal that will amount to \$500 in 6 years at $4\frac{1}{2}$ per cent compound interest.

6. Find the principal that will amount to \$1200 in 9 years at 5 per cent compound interest.

7. Find the principal that will amount to \$750 in 8 years at 5½ per cent compound interest.

8. Find the rate if \$648 amounts to \$788.38 in 5 years.

9. Find the rate if \$9660 amounts to \$12945.40 in 6 years.

10. In what time will \$4450 amount to \$6450 at 3½ per cent compound interest?

REVIEW EXERCISE XIX

Compute by logarithms:

1. $\dfrac{.076917 \times 26.3}{.5478 \times 3120.7}$.

2. $\dfrac{10000 \times .38057}{0.0098076}$.

3. $\sqrt[3]{\dfrac{-.08765 \times 72.09^2}{.0001 \times 890000}}$.

4. $\dfrac{480.63 \times .0856}{256 \times \sqrt{.843}}$.

5. $\dfrac{3.1416 \times \sqrt{468.5}}{\sqrt[4]{.9635}}$.

6. $\sqrt[3]{\dfrac{92.3 \times .08763}{.003151 \times 5\sqrt{30}}}$.

7. $\dfrac{(.573)^{\frac{3}{5}}}{8693.8 \times \sqrt[4]{.3307}}$.

8. $\sqrt{\dfrac{1.76 \times 43.278 \times (\frac{3}{2})^3}{4896.7 \times .03645}}$.

9. $2^{\frac{1}{3}} \times \left(\dfrac{1}{2}\right)^{\frac{1}{4}} \times \sqrt[3]{\dfrac{3}{2}} \times \sqrt{.1}$.

10. $\sqrt[3]{\dfrac{ar^2l}{mn^4}}$ when $\begin{cases} a = .00864 \\ r = 1.8769 \\ l = 300760 \\ m = 10000 \\ n = 2.66. \end{cases}$

Solve:

11. $28.8^x = 12750$.

12. $2^{2x+3} = 6^{x-1}$.

13. $2^{x+1} = .5^{2x-5}$.

14. $\log_x 32 = 5$.

15. $x = \log_{16} 128$.

16. $3^{2x-2} = (9^{-1})^{x-3}$.

17. $(\frac{1}{2})^{3x} = 32^{-3}$.

18. $2^{3x+1} = 7^{2x-1}$

19. $a^x = b^{2x}c^5$.

20. $x = \log_{81} 27$.

21. $(1.025)^{3x} = 1.01$.

22. $64^x = \frac{1}{2}$.

23. Given, $\begin{cases} g = 980.18 \\ l = 864.12 \\ \pi = 3.1416. \end{cases}$

Find t from the formula, $t = 2\pi\sqrt{\dfrac{l}{g}}$

24. Given $\log 2 = 0.3010$, find, without table, $\log \sqrt{5}$.

25. When $a = 1.05$, $b = .36$, find x,

$$x = \frac{\sqrt[3]{b^2(a+b)}}{ab}.$$

26. Find the amount of $625.75 for 5 years at 3 per cent compound interest.

27. Find the amount of $1000 for 5 years at 4 per cent compounded semi-annually.

CHAPTER XX

BINOMIAL THEOREM

158. Binomial Theorem for Positive Integral Exponents.

By actual multiplication, we find, that

$$(a+b)^2 = a^2 + 2ab + b^2$$
$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$
$$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$

To show the law according to which these various terms are formed, these expansions may be written:

$$(a+b)^2 = a^2 + 2ab + \frac{2\cdot 1}{1\cdot 2}b^2$$
$$(a+b)^3 = a^3 + 3a^2b + \frac{3\cdot 2}{1\cdot 2}ab^2 + \frac{3\cdot 2\cdot 1}{1\cdot 2\cdot 3}b^3$$
$$(a+b)^4 = a^4 + 4a^3b + \frac{4\cdot 3}{1\cdot 2}a^2b^2 + \frac{4\cdot 3\cdot 2}{1\cdot 2\cdot 3}ab^3 + \frac{4\cdot 3\cdot 2\cdot 1}{1\cdot 2\cdot 3\cdot 4}b^4$$

Examination shows the following facts:

1. The number of terms is one more than the exponent of the binomial.

2. The exponent of the first term is equal to the exponent of the binomial.

3. The exponents of a decrease by 1 in successive terms while the exponents of b increase by 1.

4. The sum of the exponents in any term is equal to the exponent of the binomial.

5. The coefficient of the first term is 1, of the second term is equal to the exponent of the first term and thereafter the coefficients are fractions, the numerator consisting of the product of factors decreasing by 1 and the denominator of the same number of factors, starting with 1 and increasing by 1.

If the exponent of the binomial is represented by n, the general form of the expansion is,

$$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{1\cdot 2}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}a^{n-3}b^3 + \dots$$

(The proof, that this general form is true for all positive integral exponents, does not come within the province of Elementary Algebra.)

This formula expresses the *Binomial Theorem* and furnishes us a convenient model for writing the expansion of any binomial.

In actual practice, it is unnecessary to write the coefficients in the same form as that given in the formula.

It is evident that any coefficient may be found by *multiplying the coefficient of the preceding term by the exponent of a and dividing by* 1 *more than the exponent of b.*

1. Expand $(a+b)^6$.

$$(a+b)^6 = a^6 + 6a^5b + \frac{6\times 5}{2}a^4b^2 + \frac{15\times 4}{3}a^3b^3 + \frac{20\times 3}{4}a^2b^4 + \frac{15\times 2}{5}ab^5 + \frac{6}{6}b^6 = a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6.$$

2. Expand $(a-b)^5$.

If $a-b$ is written $a+(-b)$, every term containing the odd power of $-b$ will be negative.

Hence $(a-b)^5 = a^5 - 5a^4b + 10a^3b^2 - 10a^2b^3 + 5ab^4 - b^5$

3. Expand $(2x^2 - 3yz)^4$.

Treating $2x^2$ as a and $-3yz$ as b,

$$\begin{aligned}(2x^2-3yz)^4 &= (2x^2)^4 + 4(2x^2)^3(-3yz) + 6(2x^2)^2(-3yz)^2 + 4(2x^2)(-3yz)^3 + (-3yz)^4\\ &= 16x^8 - 96x^6yz + 216x^4y^2z^2 - 216x^2y^3z^3 + 81y^4z^4\end{aligned}$$

4. Expand $(1 - x + x^2)^3$.

Write the trinomial as a binomial.

$$\begin{aligned}[(1-x)+x^2]^3 &= (1-x)^3 + 3(1-x)^2(x^2) + 3(1-x)(x^2)^2 + (x^2)^3\\ &= 1 - 3x + 3x^2 - x^3 + 3x^2(1-2x+x^2) + 3x^4(1-x) + x^6\\ &= 1 - 3x + 3x^2 - x^3 + 3x^2 - 6x^3 + 3x^4 + 3x^4 - 3x^5 + x^6\\ &= 1 - 3x + 6x^2 - 7x^3 + 6x^4 - 3x^5 + x^6\end{aligned}$$

EXERCISE 118

Expand:

1. $(x + y)^7$.
2. $(x - 2y)^6$.
3. $(2x + y^2)^4$.
4. $(a^3 - b^3)^5$.
5. $\left(\frac{x}{y} - \frac{y}{x}\right)^4$.
6. $(1 - 2a)^6$.
7. $(a^{\frac{1}{2}} + b^{\frac{1}{3}})^5$.
8. $(x^{-1} - y^2)^7$.
9. $(a^{-\frac{1}{2}} + 2b^{-3})^4$.
10. $(\sqrt{a} - \sqrt[3]{b})^5$.
11. $(3x^2 + 2y^3)^6$.
12. $(a - b + c)^4$.

TO FIND ANY REQUIRED TERM

159. Suppose that the number of the required term is r. From the binomial formula we see that any particular term, as the 4th term $\frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} a^{n-3}b^3$, has for its coefficient a fraction, whose numerator contains the factors $n(n-1)(n-2)$, that is, the factors decrease by 1 until we reach the factor $n - (r - 2)$ or $n - r + 2$. The denominator consists of the factors from 1 up to $r - 1$. The coefficient of the rth term may then be written

$$\frac{n(n-1)(n-2) \ldots . (n-r+2)}{1 \cdot 2 \cdot 3 \ldots . r-1}.$$

The exponent of a is $n - (r - 1)$ or $n - r + 1$ and the exponent of b is $r - 1$.

The product of a number of factors from 1 up to some other number, as 5, that is $1 \cdot 2 \cdot 3 \cdot 4 \cdot 5$, is called *factorial* 5 and is written $\underline{|5}$.

Hence the rth term of the binomial formula may be written

$$\frac{n(n-1)(n-2) \ldots . (n-r+2)}{\underline{|r-1}} a^{n-r+1}b^{r-1}$$

where r represents the number of the required term.

1. Find the 6th term of the expansion $\left(\frac{x^2}{y} - \frac{\sqrt{y}}{2x}\right)^7$.

$$\left(\frac{x^2}{y} - \frac{\sqrt{y}}{2x}\right)^7 = (x^2y^{-1} - 2^{-1}x^{-1}y^{\frac{1}{2}})^7$$

Here $n = 7$ and $r = 6$.

Use the formula $\frac{n(n-1)(n-2)\;.\;.\;.\;.\;(n-r+2)}{\underline{|r-1}}a^{n-r+1}b^{r-1}$.

Substitute the values of n, r, a, and b.

$$\frac{7\times6\times5\times4\times3}{\underline{|5}}(x^2y^{-1})^2(-2^{-1}x^{-1}y^{\frac{1}{2}})^5 =$$

$$\frac{7\times6\times5\times4\times3}{1\times2\times3\times4\times5}x^4y^{-2}\times-2^{-5}x^{-5}y^{\frac{5}{2}} = -\tfrac{21}{32}x^{-1}y^{\frac{1}{2}}.$$

2. Find the term containing a^8 in the expansion $(2a^2 - a^{\frac{1}{2}})^7$.

To find the number of the term containing a^8, first disregard the coefficient in the formula.

Using, $a^{n-r+1}b^{r-1}$, substituting the values of a, b, and n we have

$$(2a^2)^{7-r+1}(-a^{\frac{1}{2}})^{r-1} = a^8$$

$$(2a^2)^{8-r}(-a^{\frac{1}{2}})^{r-1} = a^8$$

neglecting the coefficient of a^2 and the sign of $a^{\frac{1}{2}}$, as they will not affect the value of the exponents,

$$a^{16-2r}\times a^{\frac{r-1}{2}} = a^8$$

$$a^{16-2r+\frac{r-1}{2}} = a^8$$

But the exponents must be equal, hence

$$16 - 2r + \frac{r-1}{2} = 8$$

$$32 - 4r + r - 1 = 16$$

$$-3r = -15$$

$$r = 5$$

Therefore the 5th term will contain a^8.

Substitute in the formula,

$$\frac{n(n-1)\;.\;.\;.\;.(n-r+2)}{\underline{|r-1}}a^{n-r+1}b^{r-1} = \frac{7\times6\times5\times4}{1\times2\times3\times4}(2a^2)^3(-a^{\frac{1}{2}})^4$$

$$= \frac{7\times6\times5\times4\times8}{1\;2\times3\times4\times}a^8 = 280a^8.$$

EXERCISE 119

1. Find the fourth term of $(2x - 3y)^6$.
2. Find the third term of $\left(\frac{a}{b} - \frac{b}{a}\right)^4$.
3. Find the fifth term of $(1 - 2x)^6$.
4. Find the fourth term of $(x^{\frac{1}{2}} + y^{\frac{1}{2}})^5$.

5. Find the eighth term of $(a^{-1} - b^{-2})^{10}$.
6. Find the seventh term of $(x^{-\frac{1}{2}} + 2y^{-3})^9$.
7. Find the sixth term of $(\sqrt{x} - \sqrt[3]{y})^8$.
8. Find the tenth term of $(3a^2 - 2b^3)^{12}$.
9. Find the term containing x^{15}, in $(3x^2 - x)^{11}$.
10. Find the term containing x^9, in $(x^3 + x)^5$.

BINOMIAL THEOREM WITH ANY EXPONENT

160. A proof, similar to the one showing the binomial formula true for any positive intègral exponent, shows that this formula will give the expansion of a binomial with any exponent, positive or negative, integral or fractional.

1. Since, $(a + b)^{-3} = \dfrac{1}{(a + b)^3}$, the value of this expansion represents the quotient obtained when 1 is divided by $(a + b)^3$. Such a division does not come out even and may be carried to any number of terms. Similarly, if $(a + b)^{-3}$ is expanded by the binomial theorem, we may obtain any required number of terms.

Thus,
$$\begin{aligned}(a + b)^{-3} &= a^{-3} + (-3)a^{-4}b + 6a^{-5}b^2 + (-10)a^{-6}b^3 + \ldots\\ &= a^{-3} - 3a^{-4}b + 6a^{-5}b^2 - 10a^{-6}b^3 + \ldots\\ &= \frac{1}{a^3} - \frac{3b}{a^4} + \frac{6b^2}{a^5} - \frac{10b^3}{a^6} + \ldots\end{aligned}$$

2. In the same way, $(a - b)^{\frac{2}{3}} = \sqrt[3]{(a - b)^2}$ and as the cube root of a quantity, not a cube, may be carried out to any number of terms, so the expansion $(a - b)^{\frac{2}{3}}$ may be carried to any required number of terms. Thus,

$$\begin{aligned}(a - b)^{\frac{2}{3}} &= a^{\frac{2}{3}} + \tfrac{2}{3}a^{-\frac{1}{3}}(-b) + (-\tfrac{1}{9})a^{-\frac{4}{3}}(-b)^2 + \tfrac{4}{81}a^{-\frac{7}{3}}(-b)^3 + \ldots\\ &= a^{\frac{2}{3}} - \tfrac{2}{3}a^{-\frac{1}{3}}b - \tfrac{1}{9}a^{-\frac{4}{3}}b^2 - \tfrac{4}{81}a^{-\frac{7}{3}}b^3 + \ldots\end{aligned}$$

Note, that to obtain the proper algebraic signs of the terms, the coefficients should be kept separate by enclosing them in parentheses, and similarly, the second term of the binomial, if negative, as shown in the above expansions.

3. Find the fifth term of $\left(x - \dfrac{3}{2\sqrt{y}}\right)^{-\frac{1}{2}}$.

Using the formula for the rth term,

$$\frac{n(n-1)(n-2)\ldots(n-r+2)}{\underline{|r-1}}a^{n-r+1}b^{r-1}$$

here $a = x$, $b = -\frac{3}{2\sqrt{y}} = -\frac{3}{2y^{\frac{1}{2}}} = -\frac{3}{2}y^{-\frac{1}{2}}$, $n = -\frac{2}{3}$, $r = 5$.

Substituting in the formula, since $n - r + 2 = -\frac{2}{3} - 5 + 2 = -\frac{11}{3}$.

$$\frac{(-\frac{2}{3})(-\frac{5}{3})(-\frac{8}{3})(-\frac{11}{3})}{1\cdot2\cdot3\cdot4}x^{-\frac{14}{3}}(-\tfrac{3}{2}y^{-\frac{1}{2}})^4 =$$

$$\frac{2\cdot5\cdot8\cdot11}{2\cdot3\cdot4\cdot3^4}x^{-\frac{14}{3}}\cdot\frac{3^4}{2^4}y^{-2} = \frac{55}{24}x^{-\frac{14}{3}}y^{-2} = \frac{55}{24x^{\frac{14}{3}}y^2}.$$

4. A root may be extracted by using the Binomial Formula.

$$\sqrt[5]{31} = \sqrt[5]{32 - 1} = \sqrt[5]{32(1 - \tfrac{1}{32})} = 2\sqrt[5]{1 - \tfrac{1}{32}} = 2(1 - \tfrac{1}{32})^{\frac{1}{5}}.$$

Expanding $(1 - \frac{1}{32})^{\frac{1}{5}}$, the number of terms to be found will depend upon the required accuracy. Suppose the fifth root is to be found to six decimals.

$$\left(1 - \frac{1}{32}\right)^{\frac{1}{5}} = 1^{\frac{1}{5}} + \frac{1}{5}(1)^{-\frac{4}{5}}\left(-\frac{1}{32}\right) + \left(-\frac{2}{25}\right)(1)^{-\frac{9}{5}}\left(-\frac{1}{32}\right)^2 +$$
$$\left(\frac{6}{125}\right)(1)^{-\frac{14}{5}}\left(-\frac{1}{32}\right)^3 +$$
$$= 1 - \frac{1}{160} - \frac{1}{12{,}800} - \frac{3}{2{,}048{,}000}.$$

As the fourth term affects only the 6th place of decimals, no additional terms need be found.

$$2\left(1 - \frac{1}{32}\right)^{\frac{1}{5}} = 2 - \frac{1}{80} - \frac{1}{6{,}400} - \frac{3}{1{,}024{,}000}$$
$$= 2 - .0125 - .000156 - .000002$$
$$\therefore \sqrt[5]{31} = 1.987342.$$

EXERCISE 120

Expand:

1. $(1 + a)^{\frac{1}{3}}$.
2. $(1 - a)^{-3}$.
3. $(1 + a)^{-\frac{2}{3}}$.
4. $(1 - a)^{\frac{3}{5}}$.
5. $(1 + 2x)^{-2}$.
6. $(2a - 3b)^{\frac{1}{4}}$.
7. $(2a + 3b)^{-\frac{1}{4}}$.
8. $\frac{1}{\sqrt[3]{x^2 - y}}$.
9. Find 4th term of $(x - 3\sqrt{a})^{-\frac{1}{2}}$.
10. Find 5th term of $\frac{1}{\sqrt[3]{a + 2b}}$.
11. Find 7th term of $(1 + x)^{-5}$.

12. Find 3d term of $\left(2 - \frac{1}{a}\right)^{-\frac{1}{3}}$.
13. Find 6th term of $(1 - 5a)^{-\frac{7}{5}}$.
14. Find $\sqrt[4]{17}$ to 5 decimals.
15. Find $\sqrt[3]{26}$ to 5 decimals.

REVIEW EXERCISE XX

Expand:

1. $(\sqrt{a^3} - \sqrt{b^3})^5$.
2. $(a^{-\frac{1}{2}} + 2x)^{\frac{1}{2}}$ to 5 terms.
3. $\left(\frac{a}{\sqrt{-2}} - 2a^{\frac{1}{2}}\sqrt{-3}\right)^3$.
4. $(2x - 3y)^{-\frac{1}{2}}$ to 5 terms.
5. $(a^{\frac{3}{2}} - b^{-2})^4$.
6. $\frac{2y}{(x - 2\sqrt{xy})^3}$ to 4 terms.
7. $\frac{1}{a - \sqrt[3]{ax}}$ to 7 terms.
8. $\left(\frac{x}{2y} - \sqrt{xy}\right)^5$.
9. $\left(2 - \frac{3a}{4}\right)^5$.
10. $\sqrt{a + x}$ to 5 terms.
11. Write the middle term of $(1 + x)^8$.
12. Find the 6th term of $\left(\frac{x}{2} - \frac{2}{3\sqrt{xy}}\right)^9$.
13. Find the 10th term of $\left(2a^2 - \frac{a}{2}\right)^{20}$.
14. Find the 5th term of $\left(3x^{-2} - \frac{2}{x}\right)^{-10}$.
15. Find the middle term of $\left(\frac{m}{\sqrt{-1}} - m^{-1}\sqrt{-1}\right)^{10}$.
16. Find the 10th term only of $(\sqrt[3]{x} - 4\sqrt{a})^{-\frac{1}{2}}$.
17. Find the term containing $x^{-\frac{5}{2}}$ in $(\sqrt[3]{x} - 4\sqrt{a})^{\frac{1}{2}}$.
18. Find the term containing $x^{\frac{13}{3}}$ in $(\sqrt{x} - \sqrt[3]{xy})^{11}$.
19. Expand $(1 + 2x - x^2)^4$.

CHAPTER XXI

INEQUALITIES

161. Problems presented so far have been solved by statements of equality between certain numbers, but some problems may contain conditions that lead to statements that one number is greater or less than another. It is the purpose of this chapter to show how such statements may yield solutions of the problems.

An **inequality** is an algebraic statement that one number is greater or less than another.

One number is *greater than another* when the remainder obtained by subtracting the second from the first is *positive,* and is *less than another* when the remainder obtained by subtracting the second from the first is *negative.* Thus, if $a - b$ is a positive number, a is greater than b; but if $a - b$ is a negative number, a is less than b.

In the scale of numbers, $\ldots +4, +3, +2, +1, 0, -1, -2$ $-3, \ldots$, any negative number is less than 0 and of two negative numbers, the one farther away from 0 is the less. Thus, -1 is less than 0 and -4 is less than -3.

The **sign of inequality** is $>$ or $<$.

It is placed between two unequal numbers always *pointing towards the smaller.*

Thus $a > b$, is read, *a is greater than b* and $a < b$ is read, *a is less than b.*

(In this chapter all letters represent real positive numbers unless otherwise specified.)

The expression on the left of the sign of inequality is called the *first member,* and that on the right, the *second member* of the inequality.

The signs $\ngtr$ and $\nless$ are sometimes used, and are read respectively, *is not greater than* and *is not less than.*

If two inequalities have their signs pointing in the same direction they are said to *subsist in the same sense.*

As, $a > b$ and $c > d$.

If the signs point in opposite directions, the two inequalities are said to *subsist in a contrary sense.*

As, $x > y$ and $m < n$.

162. In many respects, an inequality may be treated like an equation.

$$\begin{array}{rcc} \text{If} & 5 > 3 \\ \text{Add} & 3 \quad\; 3 \\ \hline & 8 > 6 \end{array} \qquad \begin{array}{rcc} \text{If} & 5 > 3 \\ \text{Subtract} & 2 \quad\; 2 \\ \hline & 3 > 1 \end{array} \qquad \begin{array}{rcc} \text{If} & 5 > 3 \\ \text{Multiply} & 2 \quad\; 2 \\ \hline & 10 > 6 \end{array}$$

$$\begin{array}{rc} \text{If} & 14 > 10 \\ \text{Divide by} & 2 \quad\;\; 2 \\ \hline & 7 > 5 \end{array}$$

Hence,

Principle 1.—*If the same number or equal numbers be added to or subtracted from both members of an inequality, the resulting inequality will subsist in the same sense.*

Principle 2.—*If both members of an inequality are multiplied or divided by the same number, the resulting inequality will subsist in the same sense.*

As the transposition of a term is equivalent to adding or subtracting the same number, hence

Principle 3.—*A term may be transposed from one side of the inequality to the other, provided its sign is changed.*

$$\begin{array}{lc} \text{If} & 5 > 3 \\ \text{And} & 7 > 5 \\ \hline \text{Add} & 12 > 8 \end{array} \text{ Hence,}$$

Principle 4.—*If the corresponding members of any number of inequalities subsisting in the same sense are added, the resulting inequality will subsist in the same sense.*

$$\begin{array}{rc} \text{If} & 5 > 3 \\ \text{And} & 3 > 2 \\ \hline \text{Multiply,} & 15 > 6 \end{array} \text{ Hence,}$$

Principle 5.—*If the corresponding members of two inequalities subsisting in the same sense are multiplied to-*

gether, the resulting inequality will subsist in the same sense, provided all the members are positive.

Since $+5 > +3$ but $-5 < -3$.

Principle 6.—*If the signs of all the terms of an inequality are changed, the resulting inequality will subsist in a contrary sense.*

Hence, any operation that changes the signs of the terms will reverse the sign of inequality.

$$\begin{array}{lrcr} \text{If} & 5 & > & 3 \\ \text{Multiply} & -2 & & -2 \\ \hline & -10 & < & -6 \end{array} \qquad \begin{array}{lrcr} \text{If} & 50 & > & 30 \\ \text{Divide by} & -10 & & -10 \\ \hline & -5 & < & -3 \end{array}$$

Principle 7.—*If both members of an inequality are multiplied or divided by the same negative number, the resulting inequality will subsist in a contrary sense.*

If two unequal numbers are subtracted from two equal numbers, the subtraction of the greater number will leave a smaller number.

$$\begin{array}{lrcr} \text{Thus,} & 10 & = & 10 \\ \text{Subtract} & 5 & > & 3 \\ \hline & 5 & < & 7 \end{array} \qquad \begin{array}{lrcr} \text{Also} & 10 & = & 10 \\ \text{Subtract} & -5 & < & -3 \\ \hline & 15 & > & 13 \end{array}$$

Principle 8.—*If each member of an inequality is subtracted from the corresponding member of an equation, the resulting inequality will subsist in a contrary sense.*

In the following operations, the result may have one member *greater than*, *equal to*, or *less than* the second member.

Sum of two inequalities subsisting in a contrary sense.

$$\begin{array}{lrcr} & 14 & > & 7 \\ \text{Add} & -7 & < & -3 \\ \hline & 7 & > & 4 \end{array} \qquad \begin{array}{lrcr} & 14 & > & 7 \\ \text{Add} & -9 & < & -2 \\ \hline & 5 & = & 5 \end{array} \qquad \begin{array}{lrcr} & 14 & > & 7 \\ \text{Add} & -10 & < & -2 \\ \hline & 4 & < & 5 \end{array}$$

Difference of two inequalities subsisting in the same sense.

$$\begin{array}{lrcr} & 14 & > & 7 \\ \text{Subtract} & 7 & > & 3 \\ \hline & 7 & > & 4 \end{array} \qquad \begin{array}{lrcr} & 14 & > & 7 \\ \text{Subtract} & 10 & > & 3 \\ \hline & 4 & = & 4 \end{array} \qquad \begin{array}{lrcr} & 14 & > & 7 \\ \text{Subtract} & 10 & > & 2 \\ \hline & 4 & < & 5 \end{array}$$

Product of two inequalities when some of the members are negative.

$$\begin{array}{rr} & 14 > 6 \\ \text{Multiply by} & -3 > -8 \\ \hline & -42 > -48 \end{array} \qquad \begin{array}{rr} & 14 > 6 \\ \text{Multiply by} & -3 > -7 \\ \hline & -42 = -42 \end{array}$$

$$\begin{array}{rr} & 14 > 6 \\ \text{Multiply by} & -3 > -\ 5 \\ \hline & -42 < -30 \end{array}$$

Quotient of two inequalities.

$$\begin{array}{rr} & 14 > 6 \\ \text{Divide by} & 2 > 1 \\ \hline & 7 > 6 \end{array} \quad \begin{array}{rr} & 14 > 6 \\ \text{Divide by} & 7 > 3 \\ \hline & 2 = 2 \end{array} \quad \begin{array}{rr} & 14 > 6 \\ \text{Divide by} & 7 > 2 \\ \hline & 2 < 3 \end{array}$$

EXERCISE 121

1. Find the limit of x when $2x - 3 > \frac{3x}{4} - \frac{3}{5}$.

Clear of fractions by multiplying by 20,

$$\begin{aligned} 40x - 60 &> 15x - 12 \\ \text{Transpose,} \quad 40x - 15x &> 60 - 12 \\ 25x &> 48 \\ x &> \frac{48}{25} \end{aligned}$$

That is, when x has any value greater than $1\frac{23}{25}$ the inequality is satisfied. $1\frac{23}{25}$ is called the *inferior limit* of x.

2. Find the limits of x when $3x - 4 > x + 2$ (1)
and $2x + 3 < x + 10$ (2)

Transpose in ①, $2x > 6$
Divide by 2, $x > 3$
Transpose in ②, $x < 7$

Therefore the values of x lie between 3 and 7.
The *inferior limit* of x is 3, and the *superior limit* is 7.

Find the limits of x:

3. $5x - 6 > 9$.
4. $4x + 6 < 7x - 1$.
5. $9x - 2 < 3x + 10$.
6. $3x - \frac{2}{3} > \frac{x}{4} - \frac{1}{6}$.
7. $\begin{cases} 7x - 3 > 2x + \frac{1}{2}. \\ 3x - 5 < x + \frac{3}{4}. \end{cases}$
8. $\begin{cases} 5 - 2x > \frac{x}{2} - 10 \\ 4x - 2 > 7 - \frac{x}{3}. \end{cases}$
9. $\begin{cases} 10 - 3x > 2x - 5 \\ 2x + \frac{3}{2} < \frac{5x}{2} + 1. \end{cases}$

10. Find limits of x and y if, $\begin{cases} 2x - 3y > 15 & (1) \\ x - 5y = 2 & (2) \end{cases}$

Multiply ② by 2, $2x - 10y = 4$ ③

Subtract ③ from ①, $7y > 11$

$\therefore\ y > \frac{11}{7}$

Multiply ① by 5 and ② by 3, $10x - 15y > 75$ ④

$3x - 15y = 6$ ⑤

Subtract ⑤ from ④, $7x > 69$

$x > \frac{69}{7}$

Hence x is greater than $9\frac{6}{7}$ and y is greater than $1\frac{4}{7}$.

From ②, $x = 5y + 2$

If $y = 2$, $x = 12$, one set of integral values for x and y.

Find the limits of x and y, and, if possible, one set of integral values of the unknown numbers:

11. $\begin{cases} 3x - 2y > 10 \\ 2x + 3y = 16. \end{cases}$

12. $\begin{cases} 5x - \frac{y}{3} > 7 \\ 3x = 5y - 9. \end{cases}$

13. $\begin{cases} 2x = 5y \\ x < 3y + 1. \end{cases}$

14. $\begin{cases} \frac{3x}{4} + \frac{2y}{3} > 5 \\ 5x - 3y = 2. \end{cases}$

15. $\begin{cases} \frac{7x}{5} - 3y < 1 \\ \frac{5x}{2} + \frac{2y}{3} = \frac{23}{6}. \end{cases}$

16. $\begin{cases} 3y - 2x < -6 \\ \frac{1}{2}y + \frac{2}{7}x = 3. \end{cases}$

17. Twice the number of pupils in a class is less than three times the number diminished by 30, and three times the number of pupils increased by 10 is greater than four times the number diminished by 25. Find the number of pupils.

18. Find the smallest whole number such that $\frac{1}{3}$ of it diminished by 3 is greater than $\frac{1}{5}$ of it increased by 3.

19. Find the limits of x, when $x^2 + 5x > 14$.

Transposing, $x^2 + 5x - 14 > 0$

Factoring, $(x + 7)(x - 2) > 0$

Hence, the product $(x + 7)(x - 2)$ is positive and either both factors are positive or both factors are negative.

But $x + 7$ is positive when $x > -7$

And $x - 2$ is positive when $x > 2$

The second condition includes the first, therefore both factors are positive when $x > 2$.

Similarly, $x + 7$ is negative when $x < -7$

And $x - 2$ is negative when $x < 2$

The first condition includes the second, therefore both factors are negative when $x < -7$.

Hence x may have any value except 2 and -7 and the values between them.

Find the limits of x in each of the following:

20. $x^2 + 2x > 15$.
21. $x^2 - 3x > 10$.
22. $x^2 + 5x > -6$.
23. $x^2 + x > 6$.
24. $x^2 + 4x > 5$.
25. $x^2 + 18 > 9x$.
26. $x^2 + 3x > 18$.
27. $x^2 + 7x > 8$.
28. $(x - 3)(2 - x) > 0$.
29. $(x - 5)(3 - x) > 0$.

163. Prove, if a and b are positive and unequal, that $a^2 + b^2 > 2ab$.

Since the square of any real number is positive,

$(a - b)^2 > 0$

or $a^2 - 2ab + b^2 > 0$.

Transposing, $a^2 + b^2 > 2ab$.

EXERCISE 122

If the letters are positive and unequal numbers, prove that:

1. $a^3 + b^3 > a^2b + ab^2$.
2. $a^2 + b^2 + c^2 > ab + ac + bc$.
3. $a^3 + b^3 + c^3 > 3abc$.
4. $(a^2 + b^2)(a^4 + b^4) > (a^3 + b^3)^2$.
5. Prove that the sum of any fraction and its reciprocal is greater than 2.
6. Prove that a positive fraction is increased by adding the same positive number to each of its terms.
7. Prove that $\frac{3a}{4b} + \frac{b}{3a} > 1$ except when $3a = 2b$.
8. Which is the greater, $\frac{a + b}{2}$ or $\frac{2ab}{a + b}$.

9 Which is the greater, $\frac{a^3 + b^3}{a^2b^2}$ or $\frac{a + b}{ab}$.

REVIEW EXERCISE XXI

1. If a is positive and b is negative, find the limit of x, in

$$\frac{x}{a} - \frac{1}{b} > \frac{x}{b} - \frac{1}{a}.$$

2. Find the limits of x in, $5x - 1 < 6x + 4$.

3. Given $2x - 3 < x + 5$ and $11 + 2x < 3x + 5$, find the limits of x.

4. Find an integral value of x if, $4x - 11 > \frac{x}{3}$ and $20 - 2x > 10$.

5. Find the limits of x when, $3(x - 4) + 2 > 4(x - 3)$ and $2(x + 1) < 4(x - 7) + 3$.

6. Prove $a + b > 2\sqrt{ab}$, when a and b are positive and unequal.

7. Find the smallest integer such that $\frac{1}{3}$ of it decreased by 7 is greater than $\frac{1}{4}$ of it increased by 6.

8. Find integral values of x and y, when,

$$\begin{cases} 51 - y < 5x \\ 21 + y > 3x \end{cases}$$

CHAPTER XXII

RATIO AND PROPORTION

RATIO

164. Two numbers may be compared by dividing the first number by the second. This indicated division is called their **ratio.**

Thus, the ratio of 7 to 5 is indicated by the fraction $\frac{7}{5}$, sometimes written $7:5$, the colon being the **ratio sign.** The ratio sign is supposed to be the sign of division with the line omitted.

Two quantities of the same kind can be compared when they are *measured by a common unit.*

Thus, we may compare 3 inches to 1 foot if both quantities are measured by a common unit, either both being expressed as a number of inches, or both expressed as a number of feet, as 3 inches : 12 inches or $\frac{1}{4}$ft. : 1 ft. There can be no ratio between quantities of different kind, as 3 lbs. and 4 ft.

The ratio of two quantities is the *ratio of their numerical measures.* Thus, the ratio of 3 inches to 12 inches is the ratio of 3 to 12 or 3:12.

The first term of the ratio, which precedes the ratio sign, is called the **antecedent** and the second, which follows the sign, the **consequent.** Both terms form a **couplet.**

In the ratio $x:y$ or $\frac{x}{y}$, x is the antecedent, y the consequent and x and y form a couplet.

When the antecedent is greater than the consequent, the ratio is called a **ratio of greater inequality.**

When the antecedent is equal to the consequent, it is called a **ratio of equality.**

When the antecedent is less than the consequent, it is called a **ratio of less inequality.**

Thus, when a and b are positive numbers,
if $a > b$, $a:b$ is a ratio of greater inequality.
$a = b$, $a:b$ is a ratio of equality.
$a < b$, $a:b$ is a ratio of less inequality.

When the antecedent and consequent of a ratio are interchanged, the resulting ratio is called the **inverse** or **reciprocal ratio** of the numbers.

Thus, the inverse or reciprocal ratio of a to b is $b:a$ or $\frac{b}{a}$.

The ratio of the squares of two numbers is called their **duplicate ratio,** the ratio of their cubes, their **triplicate ratio.**

Thus, the duplicate ratio of a to b is $a^2:b^2$; the triplicate ratio is $a^3:b^3$.

165. Two quantities that can be exactly expressed in integers in terms of some common unit are said to be **commensurable numbers.**

The common unit is called a **common measure.**

Thus, $2\frac{5}{16}$ inches and $3\frac{1}{8}$ inches are commensurable, since, by using $\frac{1}{16}$ of an inch as a common measure, $2\frac{5}{16}$ inches contains 37 of this unit and $3\frac{1}{8}$ inches contains 50 of this unit.

Hence the ratio of $2\frac{5}{16}$ inches to $3\frac{1}{8}$ inches is the ratio 37:50, a **commensurable ratio.**

When two quantities cannot be exactly expressed in integers in terms of some common unit of measure, they are said to be **incommensurable.**

Thus, the diameter of a circle and its circumference are incommensurable quantities. If the diameter can be exactly expressed in terms of some linear unit, the circumference cannot be so expressed. If the diameter is 2 inches, the circumference is $2 \times 3.1416 +$, an approximate value only. These approximations may be found to any required accuracy by using the necessary number of decimal places but will never be absolutely accurate. The ratio of the diameter of a circle to its circumference is therefore an **incommensurable ratio.**

Similarly, the ratio of $\sqrt{2}$ to 3 is an incommensurable ratio, since the $\sqrt{2}$ cannot be found exactly. If it were required to find the value of this ratio, correct to $\frac{1}{1,000,000}$, the value of $\sqrt{2}$ must be carried out to six decimal places,

$$\sqrt{2} \div 3 = \frac{\sqrt{2}}{3} = \frac{1.414213+}{3}$$

PRINCIPLES OF RATIOS

166. Since every ratio is a fraction, ratios have the same properties as fractions, that is, they may be reduced to lower terms, may be added or subtracted, multiplied or divided, like fractions.

Principle 1.—*The value of a ratio is unchanged when both terms of the ratio are multiplied or divided by the same number.*

Principle 2.—*A ratio is multiplied by a number, when the antecedent is multiplied or the consequent divided, by that number.*

Principle 3.—*A ratio is divided by a number, when the antecedent is divided or the consequent multiplied by that number.*

Principle 4.—*A ratio of greater inequality is decreased and a ratio of less inequality is increased by adding the same positive number to each term.*

Suppose a, b, and c to be positive numbers, and the ratio is $\frac{a}{b}$.

1. When $a > b$, to prove that the addition of c to each term decreases the ratio, that is, $\frac{a+c}{b+c} < \frac{a}{b}$ or $\frac{a+c}{b+c} - \frac{a}{b} < 0$.

But $\frac{a+c}{b+c} - \frac{a}{b} = \frac{b(a+c) - a(b+c)}{b(b+c)} =$

$$\frac{ab + bc - ab - ac}{b(b+c)} = \frac{c(b-a)}{b(b+c)}$$

Since $a > b$, $b - a$ is negative, hence $\frac{a+c}{b+c} - \frac{a}{b}$ is negative, showing that $\frac{a+c}{b+c} < \frac{a}{b}$.

2. When $a < b$, to show that $\frac{a+c}{b+c} > \frac{a}{b}$.

Then, $\frac{a+c}{b+c} - \frac{a}{b} = \frac{c(b-a)}{b(b+c)}$ as before. But $b - a$ is positive.

Therefore, $\frac{a+c}{b+c} > \frac{a}{b}$.

Ratios are **compounded** by taking the product of the fractions that represent them.

Thus the ratio compounded of $a : b$, $c : d$, $e : f$, is $\frac{a}{b} \times \frac{c}{d} \times \frac{e}{f} = \frac{ace}{bdf}$.

EXERCISE 123

1. What is the ratio of $6x$ to $3x$; of $3x$ to $6x$; of ay to by?

2. Which is the greater ratio, $2 : 3$ or $3 : 4$?

3. What is the ratio of $\frac{1}{2}$ to $\frac{1}{3}$; of $\frac{5}{8}$ to $\frac{3}{4}$?

4. What is the inverse ratio of $3 : 5$; of $\frac{3}{4} : \frac{7}{8}$?

5. What is the duplicate ratio of $1 : 2$; the triplicate ratio of $2 : 3$?

6. What is the ratio of 5 days to 2 weeks; of 11 inches to $\frac{3}{4}$ of a yard?

7. Compound the ratios, $\frac{1}{2} : \frac{1}{3}$ and $\frac{5}{8} : \frac{3}{4}$.

8. If $a > b$, which is the greater ratio, $a + b : a - b$ or $a^2 + b^2 : a^2 - b^2$?

9. Two numbers are in the ratio of $2 : 3$. Find the ratio of the remainders when 5 is subtracted from each number.

10. Change each to a ratio whose antecedent is 1, $\frac{4}{5}x : \frac{2}{5}x^2$, $.7m^2 : 1.4m$.

11. Reduce the ratios $a : b$ and $c : d$, to ratios having the same consequent.

12. A floor has an area of 320 sq. ft. Find its dimensions, if the width is to the length in the ratio $4 : 5$.

PROPORTION

167. An equation consisting of two equal ratios is called a **proportion.** Thus if $a:b$ is equal to $c:d$, $a:b = c:d$ is a proportion, sometimes written $a:b :: c:d$, which is read, a is to b as c is to d. The four dots are supposed to represent the extremities of the two lines forming the sign of equality. It is usually more convenient to write $\frac{a}{b} = \frac{c}{d}$.

If $a:b = c:d$, the four quantities, a, b, c, d, are called **proportionals** and the four quantities are said to be in **proportion.** The first and fourth terms of the proportion are called the **extremes** and the second and third terms, the **means.**

When the proportion contains four quantities, as $a:b = c:d$, d is called the **fourth proportional** to a, b, and c.

When the proportion contains three quantities, as, $a:b = b:c$, c is called the **third proportional** to a and b, and b is called **a mean proportional** between a and c.

A series of equal ratios in which each consequent is equal to the next antecedent, is called a **continued proportion.** As, $a:b = b:c = c:d = d:e$.

Since any proportion may be expressed as a fractional equation, any changes that can be made in such an equation without destroying the equality of the two members, may also be made in the proportion, without destroying the equality of the two ratios.

PRINCIPLES OF PROPORTIONS

168. Principle 1.—*In any proportion, the product of the means equals the product of the extremes.*

If $a:b = c:d$

Then $\frac{a}{b} = \frac{c}{d}$

Clearing of fractions, $ad = bc$.

If $a:b = b:c$,

Then $b^2 = ac$

And $b = \sqrt{ac}$, or,

A mean proportional between two numbers is equal to the square root of their product.

Principle 2.—*If the product of two numbers is equal to the product of two other numbers, either pair may be made the means of a proportion and the other pair the extremes.*

If $$ad = bc$$

Dividing by bd, $$\frac{a}{b} = \frac{c}{d}$$

or $$a:b = c:d$$

Principle 3.—*If four numbers are in proportion, the ratio of the antecedents is equal to the ratio of the consequents, that is, the numbers are in proportion by* **alternation.**

If $$a:b = c:d$$

Then $$ad = bc$$

Dividing by cd, $$\frac{a}{c} = \frac{b}{d}$$

or $$a:c = b:d$$

Principle 4.—*If four numbers are in proportion, they are in proportion by* **inversion,** *that is, the ratio of the second to the first is equal to the ratio of the fourth to the third.*

If $$a:b = c:d$$

Then $$bc = ad$$

Dividing by ac, $$\frac{b}{a} = \frac{d}{c}$$

or $$b:a = d:c$$

Principle 5.—*If four numbers are in proportion, they are in proportion by* **composition,** *that is, the sum of the terms of the first ratio is to either term of the first ratio as the sum of the terms of the second ratio is to the corresponding term of the second ratio.*

If $$a:b = c:d$$

or $$\frac{a}{b} = \frac{c}{d}$$

Then $$\frac{a}{b} + 1 = \frac{c}{d} + 1$$

or $$\frac{a+b}{b} = \frac{c+d}{d}$$

If $$a : b = c : d = e : f.$$

or $$\frac{a}{b} = \frac{c}{d} = \frac{e}{f}$$

let $r =$ the value of each ratio.

then $$a = br,\ c = dr,\ e = fr.$$

Adding these equations, $a + c + e = (b + d + f)r$.

or $$\frac{a + c + e}{b + d + f} = r = \frac{a}{b} = \frac{c}{d} = \frac{e}{f}$$

Hence, $a + c + e : b + d + f = a : b$ or $c : d$ or $e : f$.

Principle 12.—*If three numbers are in continued proportion, the ratio of the extremes is equal to the square of either ratio.*

If $$a : b = b : c$$

or $$\frac{a}{b} = \frac{b}{c}$$

Multiply by $\frac{a}{b}$, $$\frac{a^2}{b^2} = \frac{a}{c}$$

But, $$\frac{a^2}{b^2} = \frac{b^2}{c^2}$$

Hence $$\frac{a}{c} = \frac{a^2}{b^2} = \frac{b^2}{c^2} \text{ or } a : c = a^2 : b^2 = b^2 : c^2$$

Principle 13.—*If four numbers are in continued proportion, the ratio of the extremes is equal to the cube of any of the given ratios.*

If $$a : b = b : c = c : d$$

or $$\frac{a}{b} = \frac{b}{c} = \frac{c}{d}$$

Then, since $$\frac{a}{b} = \frac{b}{c}$$

$$\frac{a}{b} = \frac{c}{d}$$

$$\frac{a}{b} = \frac{a}{b}$$

Multiply these three equations,

$$\frac{a^3}{b^3} = \frac{a}{d}$$

But $\frac{a^3}{b^3} = \frac{b^3}{c^3} = \frac{c^3}{d^3}$ hence $\frac{a}{d} = \frac{a^3}{b^3} = \frac{b^3}{c^3} = \frac{c^3}{d^3}$

or, $a : d = a^3 : b^3 = b^3 : c^3 = c^3 : d^3$.

EXERCISE 124

1. Find the value of x, when $3 : x = 7 : 42$.

If $$3 : x = 7 : 42$$

Product of the means equals the product of the extremes,

$$7x = 126$$

$$\therefore x = 18$$

Find x in the following proportions:

2. $5 : x = 4 : 3$.
3. $x : 1 = 9 : 3$.
4. $3 : x = x : 12$.
5. $x + 1 : x = 5 : 4$.
6. $x : x - 1 = 12 : 10$.
7. $x + 3 : x - 3 = 20 : 10$.
8. Find the mean proportional between 2 and 32.
9. Find the mean proportional between $\sqrt{2}$ and $\sqrt{18}$.
10. Find the third proportional to 3 and 5.
11. Find the fourth proportional to 4, 6, and 11
12. If $a : b = c : d$, prove that $3a + 4b : 4a - 3b = 3c + 4d : 4c - 3d$.

Since, $a:b = c:d$

Then $3a:b = 3c:d$

And $3a:4b = 3c:4d$

By composition, $3a + 4b:4b = 3c + 4d:4d$ ①

Also since, $a:b = c:d$

$4a:b = 4c:d$

And $4a:3b = 4c:3d$

By division, $4a - 3b:3b = 4c - 3d:3d$ ②

From ①, $3a + 4b:3c + 4d = b:d$

From ②, $4a - 3b:4c - 3d = b:d$

Hence $3a + 4b:3c + 4d = 4a - 3b:4c - 3d$

By alternation, $3a + 4b:4a - 3b = 3c + 4d:4c - 3d$

When $a:b = c:d$, show that,

13. $c:a = d:b$.
14. $ac:c^2 = bd:d^2$.
15. $a^2:ab = c^2:cd$.
16. $\frac{a}{2}:mb = \frac{c}{2}:md$.

17. $2a - b : 2a + b = 2c - d : 2c + d$.

18. Solve for x, $\dfrac{\sqrt{x} + \sqrt{x+5}}{\sqrt{x} + 3} = \dfrac{\sqrt{x} - \sqrt{x+5}}{\sqrt{x} - 3}$.

By applying the principles of proportion, the work of solution is greatly lessened.

By alternation, $\dfrac{\sqrt{x} + \sqrt{x+5}}{\sqrt{x} - \sqrt{x+5}} = \dfrac{\sqrt{x} + 3}{\sqrt{x} - 3}$

By composition and division, $\dfrac{2\sqrt{x}}{2\sqrt{x+5}} = \dfrac{2\sqrt{x}}{6}$

Since the antecedents are equal, the consequents are equal.

Therefore

$$\begin{aligned} 2\sqrt{x+5} &= 6 \\ \sqrt{x+5} &= 3 \\ x + 5 &= 9 \\ x &= 4 \end{aligned}$$

Solve for x by the principles of proportion:

19. $\dfrac{\sqrt{x+7} + 3}{\sqrt{x+7} - 3} = \dfrac{\sqrt{2x+9} + 2}{\sqrt{2x+9} - 2}$.

20. $\dfrac{\sqrt{x} + \sqrt{a}}{\sqrt{x} - \sqrt{a}} = \dfrac{b}{c}$.

21. $\dfrac{\sqrt{a} + \sqrt{a+x}}{\sqrt{a} - \sqrt{a+x}} = \dfrac{\sqrt{b} + \sqrt{b-x}}{\sqrt{b} - \sqrt{b-x}}$.

22. $\dfrac{\sqrt{x+2} + \sqrt{x-3}}{\sqrt{x+2} - \sqrt{x-3}} = \dfrac{\sqrt{5x+1} + \sqrt{2x+2}}{\sqrt{5x+1} - \sqrt{2x+2}}$.

23. Divide 48 into two parts so that they shall be in the ratio of 7 to 9.

24. Two numbers are in the ratio of 2 to 3. If each is diminished by 3, they will be in the ratio of 1 to 2. Find the numbers.

25. Find a number that when added to each of the numbers, 3, 5, 28 and 35, will give four numbers in proportion.

26. If an iron wire one tenth of an inch in diameter will safely support 300 pounds, how many pounds will a wire one-fifth of an inch in diameter support? (The weights are directly proportional to the squares of the diameters.)

27. A force of 100 pounds is applied to the small piston of an hydraulic press. What pressure is exerted on the larger piston if the diameters are, respectively, 1 inch and 5 inches? $\left(\frac{F_1}{F_2} = \frac{A_2}{A_1}\right)$.

REVIEW EXERCISE XXII

1. The sum of the three sides of a triangle is 240 feet, and the ratio of the sides is as 3 : 4 : 5. Find the length of each side.

2. Divide 16 into two parts such that their product is to the sum of their squares as 3 is to 10.

3. Find a mean proportional between $\frac{a+b}{a-b}$ and $\frac{a^2-b^2}{a^2b^2}$.

4. Find y, when $6y - a : 4y - b = 3y + b : 2y + a$.

5. Find a third proportional to a^2 and 7; a fourth proportional to $5ab$, $3ac$, and $5bc$.

6. Find x and y, $\begin{cases} x : y = x + y : 42 \\ x : y = x - y : 6. \end{cases}$

7. Solve, by principles of proportion,

$$\frac{\sqrt{a} + \sqrt{a+x}}{\sqrt{a} - \sqrt{a+x}} = \frac{\sqrt{b} + \sqrt{x-b}}{\sqrt{b} - \sqrt{x-b}}.$$

8. If $a : b = c : d$, show that,

$$a - 3b : a + 3b = c - 3d : c + 3d.$$

9. Solve, by principles of proportion,

$$\frac{3 + \sqrt{2x+3}}{5 - \sqrt{2x+3}} = \frac{4 + \sqrt{x+1}}{4 - \sqrt{x+1}}.$$

10. Solve, by principles of proportion,

$$\frac{x^2 - 2x + 3}{2x - 3} = \frac{x^2 - 3x + 5}{3x - 5}.$$

11. If 1 be added to each of two numbers their ratio is 1: 2. The difference of their squares is to 3 more than their product as 5: 3. Find them .

12. For what value of x will $2x - 1$ be a mean proportional between $x + 5$ and $4x - 13$?

13. Find the values of x so that, $2x - 7, 3x + 1, 4x - 3, 5(x + 1)$, may form a true proportion.

14. In a certain school the number of boys increased 10 per cent during the year, the number of girls increased 5 per cent, the total attendance increased $6\frac{2}{3}$ per cent. Find the ratio of the number of boys to the number of girls at the beginning of the year.

15. Given $\begin{cases} 5x - y + 4z = 0 \\ 2x + 5y - 3z = 0 \end{cases}$

Find the ratio of x to y and of x to z.

CHAPTER XXIII

VARIATION

169. Constants have been defined as numbers that remain the same throughout the discussion of a problem and **variables** as numbers whose values are constantly changing.

Thus, the height of a room is a constant quantity, but the distance of a moving fly from the floor is a variable.

If two variables be so related, that when one changes, the other has a corresponding change, then one is said to be a **function** of the other, that is, the one quantity depends upon the other for its value.

Thus, if a train runs at a certain speed, the distance it travels depends upon the time during which it runs, that is, the distance is a function of the time.

One quantity is said to **vary directly** as another if the ratio of the two quantities remains *constant* throughout the change.

In the preceding illustration, the distance the train travels varies directly with the time.

The **sign of variation** is $\propto$ and is read, *varies as.* Thus, $x \propto y$ is read *x varies as y* and is a brief method of writing the proportion,

$$x : y = x_1 : y_1$$

when x_1 is the value to which x has changed when y has changed to y_1.

Since the ratio of x to y remains constant when $x \propto y$, this constant ratio may be represented by k, that is, $\frac{x}{y} = k$, whence, $x = ky$.

Therefore, *if x varies as y, x is equal to a constant times y.*

170. If one quantity varies as the **reciprocal** of another, one is said to **vary inversely** as the other.

Thus, the time required to do a certain piece of work varies inversely as the number of persons employed. For if it takes 5 men 4 days to do the work, it will take 4 men 5 days, 2 men 10 days, and 1 man 20 days to do it.

Inverse variation is expressed, $x \propto \frac{1}{y}$ or the ratio of x to $\frac{1}{y}$ is constant, hence $\frac{x}{\frac{1}{y}} = k$ or $xy = k$.

Therefore, *if x varies inversely as y, their product is constant.*

One quantity **varies jointly** as two others, when it varies as their product.

Thus, the area of a rectangle depends upon its base and altitude and therefore varies jointly as its base and altitude. If the base is doubled and the altitude trebled, the area would be six times as large.

If $x \propto yz$, and the constant ratio of x to yz is k,

$$\frac{x}{yz} = k \quad \text{or} \quad x = kyz.$$

Hence, *if x varies jointly as y and z, x is equal to a constant times their product.*

One quantity **varies directly** as a second and **inversely** as a third when it varies jointly as the second and the reciprocal of the third.

Thus, the time required to do a piece of work varies *directly* as the amount of work to be done and *inversely* as the number of persons employed. If the work were 4 times as much and the number of persons twice as large, the time required to do the work would be twice as long.

If $x \propto y \cdot \frac{1}{z}$ or $x \propto \frac{y}{z}$ and $\frac{x}{\frac{y}{z}} = k$, $x = k\frac{y}{z}$.

Hence, *if x varies directly as y and inversely as z, x is equal to a constant times $\frac{y}{z}$.*

171. In the case of three related variables x, y, and z,

If x varies as y when z is constant, and x varies as z when y is constant, then x varies as yz when both y and z are variable.

Thus, the area of a rectangle varies as the altitude when the base remains constant; it varies as the base when the altitude is constant; and the area varies as the product of base and altitude when both vary.

Proof:

The variation of x depends upon the variations of y and z. Suppose the variations of y and z to take place successively. Let y change to y_1, while z remains constant, causing x to change to x_1.

Then $\frac{x}{x_1} = \frac{y}{y_1}$ ①

Now let z change to z_1 while y retains its value of y_1, causing x_1 to change to x_2.

Then $\frac{x_1}{x_2} = \frac{z}{z_1}$ ②

Multiply ① by ②, $\frac{x}{x_2} = \frac{yz}{y_1 z_1}$.

or $x = \frac{x_2}{y_1 z_1} yz.$

But as x_2, y_1 and z_1, are particular values of x, y, and z, they are constants, and $\frac{x_2}{y_1 z_1}$ is a constant which may be represented by k.

Then $x = kyz$

And $x \propto yz$

In a similar way, it may be proved that if x varies as each of three or more numbers, y, z, m, n, . . . when the others are constant, then x varies as their product when all vary.

Thus, the volume of a rectangular solid varies as the length when the breadth and thickness remain constant; it varies as the breadth when the length and thickness remain constant; it varies as the thickness when the length and breadth remain constant; and it varies as the product of length, breadth and thickness when all three vary.

EXERCISE 125

1. If x varies inversely as y, and $x = 4$ when $y = 3$, what is the value of x when $y = 6$?

It is always advisable to find the constant of variation, if possible.

Since $x \propto \frac{1}{y}$, $x = \frac{k}{y}$.

substituting the values of x and y, $4 = \frac{k}{3}$ $\therefore k = 12$.

when $y = 6$, $x = \frac{12}{6} = 2$.

2. If $x \propto \frac{1}{y}$, and $x = 10$ when $y = 5$, find the value of x when $y = 25$.

3. If $x \propto \frac{y}{z}$, and $x = 5$ when $y = 10$ and $z = 6$, what is the value of x when $y = 15$ and $z = 5$?

4. If x varies jointly as y and z and inversely as the square of v, and $x = 20$, when $y = 5$, $z = 6$ and $v = 3$, what is the value of x expressed in terms of y, z, and v?

5. If $x \propto y$ and $y \propto z$, show that $xz \propto y^2$.

If $x \propto y, x = my$ (1)

If $y \propto z$, $y = nz$ or $z = \frac{y}{n}$ (2).

Multiply (1) by (2), $xz = \frac{m}{n} y^2$.

Since $\frac{m}{n}$ is a constant, $xz \propto y^2$.

6. If $x \propto \frac{1}{y}$ and $y \propto \frac{1}{z}$, show that $x \propto z$.

7. If $x \propto y$ and $z \propto y$, show that $(x \pm z) \propto y$.

8. The circumference of a circle varies as its diameter. If the circumference of a circle, whose diameter is 1 foot, is 3.1416 feet, find the circumference of a circle 25 feet in diameter.

9. The area of a circle varies as the square of the radius. If the area of a circle whose radius is 1 foot is 3.1416 square feet, find the area of a circle whose radius is 10 feet.

10. The volume of a sphere varies as the cube of its radius. The volume of a sphere whose radius is 1 foot is 4.1888 cubic feet, find the volume of a sphere whose radius is 3 feet.

11. The velocity of a falling body varies as the time during which it has fallen from rest. If the velocity of a falling body is 96 feet per second at the end of 3 seconds, find its velocity at the end of 10 seconds.

12. The distance a body falls from rest varies as the square of the time of falling. If a body falls 144.72 ft. in 3 seconds, how far will it fall in 8 seconds?

13. The volume of a right circular cone varies jointly as its altitude and the square of the radius of its base. If the volume of a cone 7 inches high and radius of base 6 inches, is 264 cubic inches, find the volume of a cone 5 inches high and base with radius 2 inches.

14. Find the radius of a sphere whose volume is the sum of the volumes of two spheres whose radii are 3 feet and 5 feet respectively.

15. If 100 men can do a piece of work in 20 days, how long will it take 40 men to do it?

16. The weight of a body near the earth varies inversely as the square of its distance from the center of the earth. If the radius of the earth is 4000 miles, at what distance from the earth would a body weigh only half as much?

17. The illumination from a source of light varies inversely as the square of the distance from the source. How far must a photographic plate, that is 12 inches from a lamp, be moved to receive twice as much light?

18. The number of times a pendulum oscillates in a given time varies inversely as the square root of its length. If a pendulum 39.1 inches long oscillates once a second, how long must a pendulum be to oscillate three times a second?

19. The velocity acquired by a body falling from rest varies as the square root of the height. If a body, that fell from a height of 400 feet, acquires a velocity of 160 ft. per second, with what velocity will a body strike the earth after falling from a height of 1000 feet?

20. The heat loss in an electric circuit varies directly as the intensity of the current and the square of the resistance. If the heat loss is 250 watts when the current is 10 amperes and the resistance 5 ohms, find the heat loss when the current is 20 amperes and the resistance 1 ohm.

21. The variation of volume and pressure of a gas is expressed by the formula $P_1V_1 = P_2V_2$.

A certain mass of gas has a volume of 5000 liters under a pressure of one atmosphere. Find to how many atmospheres the pressure must be increased to reduce the volume to 1000 liters.

22. A steel tank having a capacity of 4 cubic feet is filled with oxygen under a pressure of 10 atmospheres. Find how much gas is in the tank measured at standard atmospheric pressure.

23. Two electric lights of 16 and 64 candle power, respectively, are placed in front of a picture so as to illuminate it equally. The 16 candle power lamp is placed 10 feet from the picture. At what distance must the other be placed?

REVIEW EXERCISE XXIII

1. If the illumination of an object at a distance of 10 feet from a source of light is 2, what is the illumination at 40 feet? (Illumination varies inversely as the square of the distance.)

2. If x varies jointly as y and z, and inversely as the square of w, and if $x = 30$ when $y = 3$, $z = 5$ and $w = 4$, what is the value of x when $y = 9$, $z = 12$, and $w = 6$?

3. The volume of gas varies as the temperature and inversely as the pressure. If the volume is 200 cubic inches when the temperature is 260° and the pressure is 15, what is the volume when the temperature is 390° and the pressure 18?

4. If y varies inversely as $x^2 - 1$, and $y = -5$ when $x = 4$, find x when $y = -15$.

5. The pressure of wind on a plane surface varies jointly as the area of surface, and the square of the wind's velocity. The pressure on a square foot is 1 pound when the wind is moving 15 miles per hour; find the velocity of the wind when the pressure on a square yard is 16 pounds.

6. If y is the sum of two quantities which vary directly as x^2 and inversely as x respectively, and $y = -\frac{1}{2}$ when $x = 1$, and $y = \frac{37}{4}$ when $x = -2$, find y when $x = -\frac{1}{2}$.

7. The distance fallen by an object from a position of rest

varies as the square of the time of falling. If a body falls $144\frac{3}{4}$ feet in 3 seconds, how far will it fall in 10 seconds?

8. The electrical resistance of a wire varies directly as the length and inversely as the area of the cross section. If the resistance of 130 yds. of wire $\frac{1}{16}$ in. in diameter is 1 ohm, what is the resistance of 100 yds. of $\frac{1}{32}$ inch wire?

9. Given that the area of a triangle varies jointly as its base and altitude, what will be the base of a triangle whose altitude is 12, equivalent to the sum of two triangles whose bases are 10 and 6, and altitudes 3 and 9 respectively?

10. If the area of a circle varies as the square of its radius, find the radius of a circle equivalent to the sum of two circles whose radii are 5 and 12 respectively.

11. The weight of a body varies inversely as the square of its distance from the center of the earth. If a body weighs 30 lbs. on the surface of the earth (4000 miles from the center), what would be the weight of the body at a distance of 24000 miles from the surface of the earth?

12. The amount of heat received from a stove varies inversely as the square of the distance from it. A person sitting 16 feet from a stove moves up to 4 feet from it. How much will this increase the amount of heat received?

13. Given that y equals the sum of three quantities which vary as x, x^2, and x^3 respectively. When $x = 1$, $y = 4$; when $x = 2$, $y = 8$; when $x = 3$, $y = 18$. Express y in terms of x.

14. The intensity of illumination from a source of light varies inversely as the square of the distance from the source. A certain incandescent light, when placed at a distance of 49 centimeters from a screen, is found to give the same illumination as a "standard" 16 candle power lamp placed at a distance of 51 centimeters from the screen. What is the candle power of the incandescent light?

15. Three spheres of lead, whose radii are 6, 8, and 10 inches respectively, are united into one. Find the radius of the resulting sphere, if the volume of a sphere varies as the cube of its radius.

16. The area of an ellipse varies jointly as its axes. The area of an ellipse is 110 square inches when the axes are 14 inches and 10 inches. Find the area when the axes are 8 and 10 inches.

17. The square of the time of a planet's revolution about the sun varies as the cube of its distance from the sun. Neptune is about thirty times as far from the sun as the earth; find the length of his year compared with ours.

CHAPTER XXIV

PROGRESSIONS

172. A **series** is a succession of numbers, each of which, after the first, is derived from the preceding number or numbers by some fixed law.

The successive numbers are called the **terms** of the series.

The first and last terms are called the **extremes,** and all the others, the **means.**

Thus, the series 2, 4, 6, 8, 10 is formed by adding 2 to each term to obtain the succeeding term.

ARITHMETICAL PROGRESSION

173. A series, each term of which, after the first, is derived from the preceding term by the addition of a constant number, is called an **arithmetical series** or **arithmetical progression.** The constant number, which is added to each term to obtain the next term, is called the **common difference,** since the difference between any term and its preceding term is constant.

Thus the series above, 2, 4, 6, 8, 10 is an arithmetical series and the common difference is 2.

A. P. is the usual abbreviation for the words *arithmetical progression.*

If the common difference is a positive number, the series is **increasing,** but if the difference is a negative number, the series is **decreasing.**

Note that the common difference is always found by subtracting one term from the succeeding term.

If a represents the first term of an A. P. and d, the common difference, the general form of such a series is, a, $a + d$,

$a + 2d$, $a + 3d$, ... where the coefficient of d is always 1 less than the number of the term.

If the number of terms is n and the last term l,

$$l = a + (n - 1)d$$

EXERCISE 126

1. Find the 9th term of the series 5, 8, 11, 14. . . . .

Consider the 9th term, the last term of the series.
The common difference is $8 - 5 = 3$.
The first term a, is 5.
Substitute these values in the formula, $l = a + (n - 1)d$.

$$l = 5 + (9 - 1)3$$
$$l = 5 + 24 = 29$$

2. Find the 10th term of the series, 3, 5, 7, 9, 11, . .
3. Find the 15th term of the series, 4, 7, 10, 13, . .
4. Find the 13th term of the series, 10, 8, 6, 4, . . .
5. Find the 20th term of the series, 1, $1\frac{1}{5}$, $1\frac{2}{5}$, $1\frac{3}{5}$, . . .
6. Find the 10th term of the series, 50, 46, 42, 38, . . .
7. Find the 12th term of the series, x, $4x$, $7x$, $10x$, . .
8. Find the 8th term of the series, $a - 4b$, $a - 2b$, a, $a + 2b$, ...
9. Find the 7th term of the series $x + 4y$, $x + 3y$, $x + 2y$, ...
10. A body, falling from rest, falls 16 feet the first second, 3 times as far during the second second, 5 times as far during the third second and so on. How far does it fall during the 6th second?

SUM OF THE SERIES

174. If a represents the first term of an A. P., d the common difference, l the last term, n the number of terms and s the sum of the n terms,

$$s = a + (a + d) + (a + 2d) + (a + 3d) + \dots + (l - d) + l$$

or $$s = l + (l - d) + (l - 2d) + (l - 3d) + \dots + (a + d) + a$$

(Writing the terms in the reverse order)

Add these two equations,

$$2s = (a + l) + (a + l) + (a + l) + (a + l) + \dots\dots + (a + l) + (a + l)$$

Since the sum of any two corresponding terms is $a + l$ and there are n terms in each series,

$$2s = n(a + l)$$

and $$s = \frac{n}{2}(a + l)$$

The two formulæ, $l = a + (n - 1)d$

and $s = \frac{n}{2}(a + l)$

involve the five numbers, a, d, l, n, s. If any *three* of these are known, the other *two* may be found from these equations.

EXERCISE 127

1. Find the sum of 15 terms of the series, $2, 2\frac{1}{4}, 2\frac{1}{2}, \dots$

Given, $a = 2$ to find s.
$d = \frac{1}{4}$
$n = 15$

As neither of the two formulæ contains the unknown and the three known numbers, we must first find l.

Using the formula, $l = a + (n - 1)d$
Substituting, $l = 2 + (15 - 1)\frac{1}{4}$
$\therefore l = 2 + \frac{7}{2} = \frac{11}{2}$

Using the formula, $s = \frac{n}{2}(a + l)$
$s = \frac{15}{2}(2 + \frac{11}{2})$
$\therefore s = \frac{15}{2} \times \frac{15}{2} = \frac{215}{4} = 53\frac{3}{4}$

Find the sum of:

2. 10 terms of the series $1, 4, 7, \dots$
3. 12 terms of the series $10, 7, 4, \dots$
4. 15 terms of the series $\frac{1}{2}, \frac{3}{4}, 1, \dots$
5. 9 terms of the series $2, 0, -2, \dots$
6. 11 terms of the series $x, 3x, 5x, \dots$
7. n terms of the series, $1, 3, 5, \dots$
8. 20 terms of the series $1, \frac{7}{8}, \frac{3}{4}, \dots$

9. A body falls 16 feet the first second, 3 times as far the second second, 5 times as far the third second and so on. How far does it fall in 6 seconds?

10. An automobile coasting down a hill, reaches the foot of the hill in 8 seconds. If it goes 10 feet the first second, 15 feet the second second, 20 feet the third second and so on, how long is the hill?

11. Given, $d = 3$, $n = 7$, $l = 11$, find a and s.

Since $l = a + (n - 1)d$

$a = l - (n - 1)d$

Substituting, $a = 11 - (7 - 1)3$

$\therefore a = 11 - 18 = -7$

Also $s = \frac{n}{2}(a + l)$

Substituting, $s = \frac{7}{2}(-7 + 11)$

$\therefore s = \frac{7}{2} \times 4 = 14$

12. Find the sum of the series, ...3, 7, ...19, if the number of terms is 14.

Given $d = 4$, $l = 19$, $n = 14$

$l = a + (n - 1)d$

or $a = l - (n - 1)d$

$a = 19 - (14 - 1)4$

$a = 19 - 52 = -33$

And $s = \frac{n}{2}(a + l)$

$s = \frac{14}{2}(-33 + 19)$

$\therefore s = 7(-14) = -98$

13. Given, a, d, s, to find l.

$$l = a + (n - 1)d \text{ ① and } s = \frac{n}{2}(a + l) \text{ ②}$$

Since n is neither given nor required, eliminate n from these two simultaneous equations.

From ②, $n = \dfrac{2s}{a + l}$

Substitute in ①, $l = a + \left(\dfrac{2s}{a + l} - 1\right) d$

Simplify, $l = a + \dfrac{(2s - a - l)d}{a + l} = a + \dfrac{2ds - ad - dl}{a + l}$

Clear of fractions, $al + l^2 = a^2 + al + 2ds - ad - dl$

Transpose, $l^2 + dl = a^2 + 2ds - ad$

Complete the square, $l^2 + dl + \left(\frac{d}{2}\right)^2 = a^2 + 2ds - ad + \frac{d^2}{4}$

$$= \frac{4a^2 + 8ds - 4ad + d^2}{4}$$

Extract square root, $l + \frac{d}{2} = \pm\frac{1}{2}\sqrt{4a^2 + 8ds - 4ad + d^2}$

$$\therefore\ l = \frac{-d \pm \sqrt{4a^2 + 8ds - 4ad + d^2}}{2}$$

14. How many terms in the series, 3, 5, 7,....71?

15. Find the sum of the series, 2, 6, 10,....62.

16. How many terms in the series 2, 5, 8, ...if the sum is 222?

17. Find the extremes of the series...27, 32, 37,..... if the sum of 17 terms is 714.

18. Find an A. P. of 10 terms, whose sum is −15, whose second term is 9 and whose sixth term is −3.

Derive the formula for:

19. l, given a, n, s.
20. s, given d, n, l.
21. a, given d, n, s.
22. d, given a, l, s.
23. n, given a, d, s.
24. Insert 4 arithmetical means between 3 and 6.

$$\left.\begin{aligned} \text{Given } a &= 3 \\ l &= 6 \\ n &= 6 \end{aligned}\right\} \quad \begin{aligned} \text{since } l &= a + (n-1)d \\ d &= \frac{l-a}{n-1} = \frac{6-3}{5} = \tfrac{3}{5} \end{aligned}$$

Hence the series is, 3, $3\frac{3}{5}$, $4\frac{1}{5}$, $4\frac{4}{5}$, $5\frac{2}{5}$, 6.

25. Insert 7 arithmetical means between 1 and 5.

26. Insert 10 arithmetical means between 20 and 3.

27. Insert 8 arithmetical means between 5 and −5.

28. Insert 5 arithmetical means between $x + 2y$ and $x - 2y$.

When three numbers are in A. P., the second term is the **arithmetical mean** between the first and third.

If the series is represented by a, A, b.

$$\begin{aligned} \text{Then } A - a &= b - A \\ \text{or } \quad 2A &= a + b \\ \text{and } \quad A &= \frac{a+b}{2}, \text{ that is,} \end{aligned}$$

The arithmetical mean between two numbers is equal to half their sum.

Find the arithmetical mean between:

29. 3 and -4.
30. $\frac{1}{2}$ and $\frac{1}{3}$.
31. $x + y$ and $x - y$.
32. $\frac{a+2b}{a-2b}$ and $\frac{a-2b}{a+2b}$.

In solving problems in A. P. involving two unknowns, the series may be represented by

$$x,\ x + y,\ x + 2y, \text{ etc.}$$

When the number of terms is small, a symmetrical form is more convenient.

Thus, a series of 3 terms, $x - y, x, x + y$, common difference y.
a series of 5 terms, $x - 2y, x - y, x, x + y, x + 2y$, common difference y.
a series of 4 terms, $x - 3y, x - y, x + y, x + 3y$, common difference $2y$.

In each case, the sum of the terms involves only one unknown.

33. The sum of three numbers in A. P. is 27 and their product is 585. Find the numbers.

Represent the series by $x - y, x, x + y$,

$$\begin{aligned} \text{Then } x - y + x + x + y &= 27 \\ 3x &= 27 \\ x &= 9 \\ \text{Also, } \quad x(x - y)(x + y) &= 585 \\ \text{substitute value of } x, \ 9(9 - y)(9 + y) &= 585 \\ \text{Divide by 9, } \quad 81 - y^2 &= 65 \\ y^2 &= 16 \\ y &= \pm 4 \end{aligned}$$

Since $x = 9$ and $y = \pm 4$,
the series is, 5, 9, 13, or 13, 9, 5.

34. The sum of three numbers in A. P. is 30 and the sum of their squares is 308. Find the numbers.

35. The sum of four numbers in A. P. is 32 and the product of the means is 8 greater than the product of the extremes. Find the series.

36. Find the sum of the even numbers 2 to 100 inclusive.

37. A common clock strikes the hours from 1 to 12. How many times does it strike in 24 hours?

38. The arithmetical mean between two numbers is 15 and the arithmetical mean between three times the first and twice the second is 35. Find the two numbers.

39. Fifteen pool balls are arranged in a triangle, so that the first row contains 1 ball, the second 2 balls, the third 3 balls and so on. How many balls in the last row?

40. A farmer objected to paying a horseshoer $1 per shoe, but agreed to pay 1 cent for the first nail, 2 cents for the second, 3 cents for the third and so on. If it takes 8 nails for each shoe, how much did the farmer gain or lose by his agreement, if he had a horse completely reshod?

GEOMETRICAL PROGRESSION

175. A series of numbers, each of which after the first, is obtained by multiplying the preceding number by a constant multiplier is called a **geometrical series** or **geometrical progression.**

Thus, 1, 2, 4, 8, 16 and x^5, x^3 x, are geometrical progressions. In the first series, the constant multiplier is 2, in the second, $\frac{1}{x^2}$.

As the constant multiplier may be found by dividing any term by its preceding term and as this division is equal to the ratio of the term to its preceding term, the constant multiplier is usually called the **common ratio** or simply **ratio.**

It is evident, that if the ratio is greater than 1, the terms of the ratio **increase** in value and when the ratio is less than 1, the terms **decrease.**

G. P. is the abbreviation for the words *geometrical progression.*

To find the *n*th or last term of a geometrical progression.

176. Let a represent the first term of a G. P., r the ratio, n the number of terms, and l the last term.

Then the series may be written in the form,

$a, ar, ar^2, ar^3, ar^4, \ldots\ldots$

where the exponent of r is always 1 less than the number of the term.

Hence the *nth* or last term, $l = ar^{n-1}$

EXERCISE 128

1. Find the 8th term of the series, 1, 2, 4, 8, ...

$a = 1$ Substitute in the formula $l = ar^{n-1}$

$r = 2$ $l = 1 \times 2^7 = 128$

$n = 8$

2. Find the 9th term of the series, 1, 3, 9,.....
3. Find the 10th term of the series $\frac{1}{8}, \frac{1}{4}, \frac{1}{2}, \ldots\ldots$
4. Find the 8th term of the series, 3, 6, 12,.....
5. Find the 7th term of the series, 2, 10, 50,.....
6. Find the 11th term of the series, 16, 8, 4,.....
7. Find the 6th term of the series $\frac{1}{3}, \frac{1}{4}, \frac{3}{16}, \ldots\ldots$
8. Find the 8th term of the series, 2, $2\sqrt{2}$, 4,.....
9. Find the 12th term of the series $x^6y, x^5y^2, x^4y^3, \ldots\ldots$

10. The population of the United States was 76 millions in 1900. If it doubles itself every 25 years, find the population in the year 2000.

12. If a rubber ball rebounds to a height equal to $\frac{3}{5}$ of its fall, find the height of its fifth rebound when dropped from a height of 50 feet.

13. A man invests $5000 at 5 per cent compound interest. What is the value of his investment at the end of 10 years?

14. The power plant installed by a traction company is valued at $50000. If it depreciates in value 10 per cent each year, what is its value at the end of 5 years?

177. A series consisting of a limited number of terms is called a **finite series**; one of an unlimited number of terms, an **infinite series.**

THE SUM OF A FINITE GEOMETRICAL SERIES

Let a = the first term
r = the ratio
n = the number of terms
l = the last term
s = the sum of the terms

Then $s = a + ar + ar^2 + ar^3 + ar^4 + \dots + ar^{n-1}$ (1)

Multiply (1) by r,

$rs = ar + ar^2 + ar^3 + ar^4 + ar^5 + \dots ar^n$ (2)

Subtract (1) from (2),

$$rs - s = ar^n - a$$

or $s(r - 1) = ar^n - a$

$$\therefore s = \frac{ar^n - a}{r - 1}$$

This formula for s is used when a, r, and n, are known.

When n is unknown but l is known,

Since $ar^{n-1} = l$

Mult. by r, $ar^n = rl$

Substituting this value in the formula for s,

$$s = \frac{rl - a}{r - 1}$$

This formula is used when $r > 1$.

Multiplying both terms of the fraction by -1,

$s = \frac{a - rl}{1 - r}$, a form more convenient when $r < 1$.

EXERCISE 129

1. Find the sum of 8 terms of the series, 1, 2, 4, . . .

$a = 1$ substituting in the formula, $s = \frac{ar^n - a}{r - 1}$
$r = 2$
$n = 8$ $s = \frac{1 \times 2^8 - 1}{1} = 255$

2. Find the sum of 9 terms of the series, 1, 3, 9, . . .
3. Find the sum of 8 terms of the series $\frac{1}{8}, \frac{1}{4}, \frac{1}{2}, \dots$
4. Find the sum of 10 terms of the series, 16, 8, 4, . . .
5. Find the sum of 6 terms of the series $\frac{1}{3}, \frac{1}{4}, \frac{8}{16}, \dots$

6. Find the sum of 5 terms of the series, $2, 2\sqrt{2}, 4, \ldots$
7. Find the sum of n terms of the series $1, 2, 4, \ldots$
8. Find the sum of n terms of the series $1, \frac{1}{2}, \frac{1}{4}, \cdots$
9. Find the sum of the series, $2, 6, 18, \ldots \ldots 1458$.
10. Find the sum of the series, $3, -\frac{3}{2}, \frac{3}{4}, \cdots\cdots -\frac{3}{128}$.

THE SUM OF AN INFINITE GEOMETRICAL SERIES

178. If the number of terms of an increasing series is unlimited, the value of the sum of the terms is unlimited and hence equal to infinity, that is, greater than any assigned value.

The sum of an infinite decreasing series may be found. Since the ratio r is less than 1, the successive terms of a G. P. become numerically less and less. Hence, in an infinite decreasing series, the last term ar^{n-1} can be made less than any assigned value but not equal to zero.

If the formula $s = \dfrac{a - rl}{1 - r}$

is written, $\qquad s = \dfrac{a}{1 - r} - \dfrac{rl}{1 - r}$

By taking a sufficient number of terms, the last term l may be made less than any assigned value, hence $\dfrac{rl}{1 - r}$ may be made so small as to be negligible.

Then, the formula for the sum of an infinite decreasing geometrical progression assumes the form

$$s = \frac{a}{1 - r}$$

EXERCISE 130

1. Find the sum of $1, \frac{1}{2}, \frac{1}{4}, \cdots$

$$a = 1 \qquad s = \frac{a}{1 - r}$$

$$r = \tfrac{1}{2} \qquad \therefore s = \frac{1}{1 - \frac{1}{2}} = \frac{1}{\frac{1}{2}} = 2$$

Find the sum of the series:

2. $1, \frac{1}{10}, \frac{1}{100}, \dots\dots$
3. $.2, .02, .002, \dots$
4. $1, -\frac{1}{2}, \frac{1}{4}, \dots$
5. $5, 1, \frac{1}{5}, \dots$
6. $50, 30, 18, \dots$
7. $x, -\frac{x}{4}, \frac{x}{16}, \dots$
8. $.1, .001, .00001, \dots$
9. Find the value of $1.666 \dots$

A decimal containing a number or group of numbers that is indefinitely repeated is called a **repeating decimal.** The value of a repeating decimal may be expressed as a common fraction by using the formula for the sum of an infinite decreasing G. P.

Since $1.666 = 1 + .6 + .06 + .006 + \dots$

Find the sum of $.6 + .06 + .006 + \dots$

Here $a = \frac{6}{10}$

$r = \frac{1}{10}$

$$s = \frac{a}{1 - r} = \frac{\frac{6}{10}}{1 - \frac{1}{10}} = \frac{6}{10} \times \frac{10}{9} = \frac{2}{3}$$

Hence $1.666+ = 1\frac{2}{3}$

Find the value of:

10. 0.333. . . . . . .
11. 1.474747. . . . .
12. 0.243243. . . . .
13. 2.8444. . . . . . .
14. 0.0303303. . . . .
15. 5.0367367. . . . .

GEOMETRICAL MEANS BETWEEN TWO TERMS

179. The name **geometrical means** is given to the terms of a G. P. lying between the extremes, that is, between the first and last terms. If the series consists of three terms, the second term is called **the geometrical mean** between the first and third terms.

If G is the geometrical mean between a and b,

then $$\frac{G}{a} = \frac{b}{G}$$

or $$G^2 = ab$$

and $$G = \pm\sqrt{ab}$$

Hence, *the geometrical mean between two numbers is equal to the square root of their product.*

Since the proportion, $\frac{G}{a} = \frac{b}{G}$, may be written by inversion, $\frac{a}{G} = \frac{G}{b}$, it follows that the *geometrical mean* between two numbers is also their *mean proportional.*

EXERCISE 131

1. Insert 4 geometrical means between 1 and 243.

$a = 1$
$l = 243$
$n = 6$ (4 means + 2 extremes)
Substituting in the formula $l = ar^{n-1}$

$$243 = 1 \times r^5$$

By inspection, since $243 = 3^5$, $3^5 = r^5$
Hence $r = 3$
And the series is 1, 3, 9, 27, 81, 243.

2. Insert 3 geometrical means between 7 and 112.
3. Insert 4 geometrical means between $\frac{1}{2}$ and $\frac{16}{243}$.
4. Insert 4 geometrical means between 2560 and $2\frac{1}{2}$.
5. Insert 3 geometrical means between 1 and 9.
6. Insert 4 geometrical means between x^5 and y^5.
7. Find the geometrical mean between:

$$\frac{x^2 + xy}{xy - 2y^2} \text{ and } \frac{xy + y^2}{x^2 - 2xy}$$

Since $G = \pm\sqrt{ab}$

$$G = \pm\sqrt{\frac{x(x + y)}{y(x - 2y)} \times \frac{y(x + y)}{x(x - 2y)}} = \pm\sqrt{\frac{(x + y)^2}{(x - 2y)^2}} = \pm\frac{x + y}{x - 2y}.$$

Find the geometrical mean between:

8. $\frac{1}{2}$ and $\frac{9}{8}$.
9. 5 and 125.

10. 243 and 3.
11. $\frac{16}{81}$ and $\frac{9}{25}$.
12. $(x - y)^2$ and $(x + y)^2$.
13. $\frac{a^2b + b^3}{a^3 - ab^2}$ and $\frac{a^3 + ab^2}{a^2b - b^3}$.

180. As in A. P., whenever three of the five elements of a G. P. are known, the two unknown elements may be found by using the two simultaneous equations given above, that is, the formulæ for l and s.

EXERCISE 132

1. Given r, n, s, to find l.

$$l = ar^{n-1} \quad ①$$

$$s = \frac{rl - a}{r - 1} \quad ②$$

From ①, $a = \frac{l}{r^{n-1}}$

Substitute in ②, $s = \frac{rl - \frac{l}{r^{n-1}}}{r - 1} = \frac{\frac{r^nl - l}{r^{n-1}}}{r - 1}.$

Multiply by $r - 1$, $s(r - 1) = \frac{r^nl - l}{r^{n-1}}$

Clear of fractions, $sr^{n-1}(r - 1) = r^nl - l = l(r^n - 1)$

$$\therefore l = \frac{sr^{n-1}(r - 1)}{r^n - 1}$$

2. Given a, r, s, to find l.
3. Given r, n, l, to find s.
4. Given r, n, s, to find a.
5. Given a, l, s, to find r.
6. If $\frac{1}{9} + \frac{1}{9}\sqrt{3} + \frac{1}{3} + \cdots\cdots = 13\frac{4}{9}(\sqrt{3} + 1)$ find the last term and the number of terms.

Here, $a = \frac{1}{9}$
$r = \sqrt{3}$
$s = 13^4(\sqrt{3} + 1)$

$$s = \frac{rl - a}{r - l}$$

Substituting, $13\frac{4}{9}(\sqrt{3} + 1) = \frac{\sqrt{3}l - \frac{1}{9}}{\sqrt{3} - 1}$

Multiply by $\sqrt{3} - 1$, $13\frac{4}{9} \times 2 = \sqrt{3}l - \frac{1}{9}$

$$26\tfrac{8}{9} = \sqrt{3}\,l - \tfrac{1}{9}$$

$$\sqrt{3}l = 27$$

$$l = \frac{27}{\sqrt{3}} = 9\sqrt{3}.$$

Since, $l = ar^{n-1}$

$$9\sqrt{3} = \tfrac{1}{9} \times (\sqrt{3})^{n-1}$$

$$81\sqrt{3} = (\sqrt{3})^{n-1}$$

$$3^{\frac{9}{2}} = (3^{\frac{1}{2}})^{n-1} = 3^{\frac{n-1}{2}}$$

By inspection, $\dfrac{n-1}{2} = \dfrac{9}{2}$

or $n-1 = 9$

$$\therefore n = 10.$$

7. If $\frac{1}{4} + \frac{1}{4}\sqrt{2} + \frac{1}{2} + \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot = \frac{15}{4}(\sqrt{2} + 1)$ find the last term and the number of terms.

8. Find the last term of the series 1, 2, 4, ... whose sum is 31.

9. Find the first term of an infinite G. P. if the ratio is $\frac{1}{5}$ and the sum $\frac{2}{3}$.

10. The sum of the G. P., 7, ... 1701, is 2548. Find the ratio.

11. The sixth term of a G. P. is 160 and the ratio is 2. Find the first and 10th terms.

12. The sum of a G. P. is 1266 greater than the first term and 844 greater than the last term. Find the ratio.

13. The population of a city increased from 30000 to 43923 in four years. Find the rate of increase.

14. A rubber ball thrown vertically to a height of 50 feet falls and rebounds 40 feet the first time, 32 feet the second time and so on. Find the whole distance through which the ball has moved before coming to rest.

15. A toboggan reaches the foot of a hill with a velocity of 200 ft. per second. How far will it go on the level if its velocity decreases each second $\frac{1}{4}$ of that of the previous second?

16. The sum of three numbers in G. P. is 26 and the sum of their squares is 364. Find the numbers.

When only three numbers are involved, they are conveniently represented by x^2, xy, y^2, or x, $\sqrt{xy}$, y, symmetrical forms.

Then $x^2 + xy + y^2 = 26$ ①

And $x^4 + x^2y^2 + y^4 = 364$ ②

Divide ② by ①, $x^2 - xy + y^2 = 14$ ③

Subtract ③ from ①, $2xy = 12$

$xy = 6$ ④

Add ④ to ①, $x^2 + 2xy + y^2 = 32$

$x + y = \pm 4\sqrt{2}$ ⑤

Subtract ④ from ③, $x^2 - 2xy + y^2 = 8$

$x - y = \pm 2\sqrt{2}$ ⑥

Add ⑤ and ⑥, $2x = \pm 6\sqrt{2}$

$x = \pm 3\sqrt{2}$

Subtract ⑥ from ⑤, $2y = \pm 2\sqrt{2}$

$y = \pm \sqrt{2}$

Hence, $x^2 = 18$, $xy = 6$, and $y^2 = 2$

17. The product of three number in G. P. is 360 and the sum of their squares is 189. Find the numbers.

18. The sum of the extremes of a G. P. is 35, and the sum of the two means is 30. Find the numbers.

Note.—Four numbers in G. P. are conveniently represented by $\frac{x^2}{y}, x, y, \frac{y^2}{x}$.

19. From a cask of wine $\frac{1}{5}$ was drawn and the cask refilled with water. What part of the contents will be wine if this is done five times?

20. The normal pressure of air is approximately 15 pounds per square inch. An air pump, used in inflating an automobile tire, increases the pressure by $\frac{1}{10}$ with each stroke. Find the pressure of the air in the tire after 5 strokes of the air pump.

HARMONICAL PROGRESSION

181. A series of numbers, whose reciprocals form an arithmetical progression, is called a **harmonical series** or **harmonical progression.**

Harmonical series is so called because musical strings of the same thickness and tension will produce harmony when their lengths are the reciprocals of the natural series of numbers, that is, 1, $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$ etc.

The general form of a harmonical progression is $\frac{1}{a}$, $\frac{1}{a+d}$, $\frac{1}{a+2d}$, and so on.

H. P. is the usual abbreviation of the words harmonical progression.

The principles of arithmetical progression are used in problems of harmonical progression since the reciprocals of the terms of any H. P. will form an A. P.

If H represents the harmonical mean between two numbers a and b,

then $\frac{1}{a}, \frac{1}{H}, \frac{1}{b}$ will form an A. P,

and $$\frac{1}{H} - \frac{1}{a} = \frac{1}{b} - \frac{1}{H}$$

or $$\frac{2}{H} = \frac{1}{a} + \frac{1}{b} = \frac{a+b}{ab}$$

$$\therefore \quad H = \frac{2ab}{a+b}$$

Hence, *the harmonical mean between two numbers is equal to twice the product of the numbers divided by their sum.*

EXERCISE 133

1. Insert 5 harmonical means between 1 and 7.

Insert 5 arithmetical means between 1 and $\frac{1}{7}$.

Then $a = 1$ $\qquad l = a + (n-1)d$

$l = \frac{1}{7}$ $\qquad d = \frac{l-a}{n-1} = \frac{\frac{1}{7} - 1}{6} = \frac{-\frac{6}{7}}{6} = -\frac{1}{7}$

$n = 7$

$A.P. = 1, \frac{6}{7}, \frac{5}{7}, \frac{4}{7}, \frac{3}{7}, \frac{2}{7}, \frac{1}{7}$

$H.P. = 1, \frac{7}{6}, \frac{7}{5}, \frac{7}{4}, \frac{7}{3}, \frac{7}{2}, 7$

2. Insert 4 harmonical means between 3 and 18.

3. Find the 6th term of the H. P., 4, 3, ...

4. The first term of a H. P. is $\frac{1}{5}$ and the third term $\frac{1}{4}$; which term will be 1?

5. Find two numbers such that one number is three times the other and their arithmetical mean exceeds their harmonical mean by $11\frac{1}{12}$.

6. Find two numbers whose difference is 6 and whose harmonical mean is $\frac{1}{6}$.

7. The arithmetical mean between two numbers exceeds

their geometrical mean by $\frac{1}{2}$, and the geometrical mean exceeds their harmonical mean by $5\frac{11}{18}$. Find the numbers.

8. Prove that $G = \pm \sqrt{A \times H}$.

9. When x, y, z, are in H. P. show that $x - y : y - z = x : z$.

10. The sixth term of a H. P. is 1 and the tenth term $\frac{5}{7}$. Find the first five terms.

REVIEW EXERCISE XXIV

1. Find the sum of the series, 1, 3, 5, . . . to n terms.

2. Three numbers, whose sum is 24, are in A. P., but if 3, 4, and 7, are added to them respectively, these sums are in G. P. Find the numbers.

3. Find the sum of eight terms of a G. P. whose last term is 1 and fifth term $\frac{1}{8}$.

4. Find the last term and the sum of the series 32, -16, $+8$, . . . to seven terms.

5. The sum of three numbers in A. P. is 27, and their product is 693. Find the numbers.

6. Insert six geometrical means between $\frac{1}{10}$ and $12\frac{4}{5}$.

7. There are four numbers, the first three of which are in G. P. and the last three in A. P. The sum of the first and last is 14 and the sum of the second and third is 12. Find them.

8. In a G. P. of ten terms, the fourth and seventh terms are -5 and 625 respectively. Find the remaining elements.

9. How many consecutive odd numbers, beginning with 7, must be taken to give 775?

10. Given $a = \frac{3}{4}$, $r = -\frac{1}{3}$, $s = \frac{91}{162}$, find l and n.

11. The arithmetical mean between two numbers is $\frac{11}{6}$, and their harmonical mean is $-\frac{40}{11}$; find the numbers.

12. Insert five harmonical means between 2 and -3.

13. The difference between two numbers is 48, and their arithmetical mean exceeds their geometrical mean by 8. Find the numbers.

14. A body, rolling down an inclined plane, goes 6 feet

the first second, 18 feet the next, 30 feet the third and so on. How many seconds will it take to roll 486 feet?

15. What number added to 1, 13, and 73, will give results in G. P.?

16. Find the sum of $\frac{n+1}{n} + \frac{n+2}{n} + \frac{n+3}{n} + \cdots$ to n terms.

17. Write three progressions of six terms, one arithmetical, one geometrical, and one harmonical, in each of which the second term is -6, and the fourth term -54.

18. Find the value of 0.11003003. . . .

19. In a "potato race," 50 potatoes are placed in a straight line 4 feet apart, the first being 5 feet from the basket. How far must a contestant travel in bringing them to the basket one at a time?

20. Find the value of 2.34848. . .

21. Find the sum of all the numbers between 100 and 600 that are divisible by 11.

22. Insert between 6 and 16, two numbers, such that the first three of the four shall be in A. P., and the last three in G. P

23. If the fourth term of a G. P. is $\frac{1}{2^4}$ and the seventh $\frac{1}{2^7}$, how many terms beginning with the first must be taken so that their sum may be $\frac{4095}{4096}$?

24. Divide $9\frac{1}{2}$ into three parts in G. P. such that the sum of the first two is to the sum of the last two as 3 is to 2.

25. Find a G. P. of which the sum of the first two terms is $2\frac{2}{3}$ and the sum to infinity $4\frac{1}{6}$.

26. In boring a well 400 feet deep, the cost is 27 cents for the first foot and an additional cent for each subsequent foot. What is the cost of boring the last foot, and the entire cost?

27. Find the sum of all even numbers from 2 to 50 inclusive.

28. Find the limiting sum of the series,
$1 + \frac{1}{5} + \frac{1}{25} + \frac{1}{125}, \cdots$ to infinity.

29. Find the value of the decimal 0.922828.....

CHAPTER XXV

SIMPLE INDETERMINATE EQUATIONS

182. A single equation involving *two* unknown numbers may have an unlimited number of solutions if no other conditions are imposed, for, if any value is assigned to one of the two unknown numbers, a corresponding value of the other may be found. Thus if, $2x + 3y = 6$, any value assigned to x will have a corresponding value of y.

Any equation that has an indefinite number of solutions is called **indeterminate.**

In the case of simultaneous equations, if the number of equations is less than the number of unknown numbers, the system will also have an indefinite number of solutions, and hence is indeterminate.

Thus the system, $\begin{cases} x + y + z = 10 \\ 2x - 3y + 4z = 20 \end{cases}$ is indeterminate.

By eliminating one of the unknowns, we obtain a single equation containing two unknowns and therefore indeterminate.

The values of the unknown numbers in an indeterminate equation are dependent upon each other, that is the values are corresponding values and must agree with the condition stated by the equation. If additional conditions are imposed upon the values of the unknowns, the number of solutions may be limited. For instance, let us impose the condition that the values of the unknowns must be *positive integers*.

1. Solve, in positive integers, $3x + 4y = 25$.

Divide the equation by the smaller coefficient, 3, and express the quotient in integers and fractions.

$$x + y + \frac{y}{3} = 8 + \frac{1}{3}$$

Transpose the fractions to the left side of the equation and the integral forms to the right.

$$\frac{y-1}{3} = 8 - x - y$$

As x and y must be positive integers, the right side of the equation has an integral value, hence the left side must also be equal to an integer.

Let m be some integer.

Then $\frac{y-1}{3} = m$

Solving for y, $y - 1 = 3m$

$y = 3m + 1.$

Substitute this value of y in the original equation.

$$3x + 4(3m + 1) = 25$$

Solve for x, $3x = 25 - 12m - 4 = 21 - 12m$

$\therefore x = 7 - 4m$

and $y = 3m + 1.$

Assign values to m to find positive integral values of x and y, when $m = 0, x = 7, y = 1$

$m = 1, x = 3, y = 4$

when $m > 1$, x becomes negative and when $m < 0$, y becomes negative; hence we have only two sets of positive integral value for x and y.

2. Solve in positive integers, $5x + 8y = 34$.

Divide by the smaller coefficient 5,

$$x + y + \frac{3y}{5} = 6 + \frac{4}{5}$$

Transpose, $\frac{3y-4}{5} = 6 - x - y$

$\frac{3y-4}{5}$ must be integral.

Let $\frac{3y-4}{5} = m$

Then $3y - 4 = 5m$

$3y = 5m + 4$

$y = \frac{5m+4}{3}$

This value of y is not an integral form because the coefficient of y in the fraction $\frac{3y-4}{5}$ is not unity.

If the fraction $\frac{3y-4}{5}$ is multiplied by an integer, it will still be equal to an integer; hence multiply the fraction by a number that will make the coefficient of y one more than a multiple of the denominator.

Multiply the fraction by 2,

$$\frac{6y - 8}{5} = y + \frac{y}{5} - 1 - \frac{3}{5}$$

Hence, to be integral, the fraction $\frac{y-3}{5}$ must be equal to some integer.

Let $\frac{y-3}{5} = m$

Then $y - 3 = 5m$

and $y = 5m + 3$

Substitute this value in the original equation,

$$5x + 8(5m + 3) = 34$$
$$5x = 34 - 40m - 24$$
$$5x = 10 - 40m$$
$$x = 2 - 8m$$
$$\text{and } y = 3 + 5m$$

When $m = 0$, $x = 2$, $y = 3$.

Any value of $m > 0$ will make x negative and any value of $m < 0$ will make y negative. Therefore, the only positive integral values are $x = 2$, $y = 3$.

EXERCISE 134

Solve in positive integers:

1. $4x + 5y = 26$.
2. $5x + 2y = 39$.
3. $3x + 5y = 50$.
4. $2x + 3y = 43$.
5. $3x + 7y = 50$.
6. $2x + y = 3$.
7. $7x + 5y = 21$.
8. $x + 2y = 9$.
9. $3x + 7y = 16$.
10. $2x + 5y = 13$.

Solve in least positive integers:

11. $5x - 2y = 15$.
12. $x - 7y = 8$.
13. $3x - 4y = 17$.
14. $7x - 2y = 69$.
15. $x - 10y = 39$.
16. $2x - 3y = 26$.

17. $\begin{cases} 4x + 3y + 2z = 25 \\ 3x - 2y + 5z = 20. \end{cases}$

18. $\begin{cases} 5x + 2y - 3z = 160 \\ 2x - 3y - 5z = 45. \end{cases}$

19. A man spent \$28 in buying shirts, some at \$2 and some at \$3. How many of each kind did he buy?

20. A coal dealer sold 10 tons of pea coal, 12 tons of nut coal and 5 tons of egg coal for \$245. The next day, at the same price, he sold 8 tons of pea, 5 tons of nut, and 10 tons of egg coal for \$204. Find the price of each.

21. In how many ways can a man pay a bill of $53 using $2 bills and $5 bills?

22. Find the smallest number that when divided by 4 or divided by 7, leaves a remainder of 3.

183. Graphical Solution of Indeterminate Equations. Any indeterminate equation of the first degree may be represented graphically by a straight line.

The positive integral values of x and y are then determined by noting at what points the line crosses the intersections of the ruled lines of the graph paper, in the *first quadrant*, since the values of x and y are both positive in the first quadrant only.

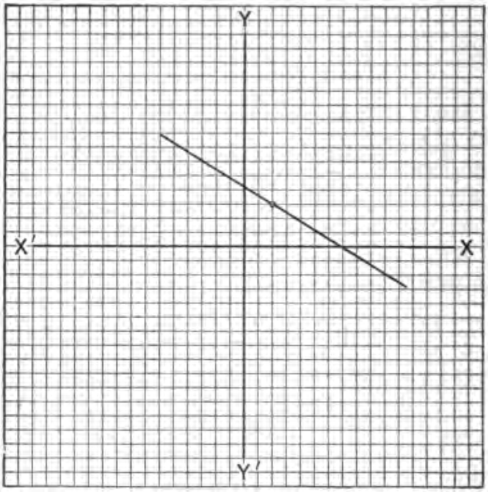


Fig. I.

Thus, the equation of Problem 2, yields the graph shown in Fig. 1. When $x = 2$, $y = 3$, the line cuts the corner of one of the squares. Since this is the only point in the *first* quadrant, the equation $5x + 8y = 34$ has only one set of positive integral values of x and y.

Note.—The success of this solution depends entirely upon the accuracy of the drawing. The line should be accurately determined by locating two points some distance apart.

The equations of Exercise 134 furnish ample material for the application of this method.

REVIEW EXERCISE XXV

1. In how many ways may \$2.75 be paid with quarters and half dollars?

2. Find the smallest number that, when divided by 5 or divided by 7, gives 4 for a remainder.

3. Solve in positive integers, $4x + 7y = 94$.

4. A farmer spends \$752 in buying horses and cows; if each horse cost \$37 and each cow \$23, how many of each kind does he buy?

5. Find two fractions whose denominators are 9 and 5 respectively, and whose sum shall be equal to $\frac{113}{45}$.

6. Solve in positive integers $\begin{cases} 5x - y + 4z = 27 \\ 2x + 3y - 5z = -8. \end{cases}$

7. A party of 20 people, consisting of men, women, and children, pay a hotel bill of \$67. Each man pays \$5, each woman \$4 and each child \$1.50. How many of each were there?

8. Solve in positive integers, $8x + 5y = 107$.

9. A woman expended 93 cents for 14 yards of cloth, some at 5, some at 7, and the rest at 10 cents a yard. How many whole yards of each did she buy?

10. What is the least number that will contain 25 with a remainder of 1, and 33 with a remainder of 2?

CHAPTER XXVI

DISCUSSION OF EQUATIONS

184. Character of the Roots of a Quadratic Equation. Every quadratic equation may be reduced to the form

$$ax^2 + bx + c = 0$$

in which a is positive, while b and c may be positive or negative.

If the roots of this general form are denoted by r_1 and r_2 then, as previously shown,

$$r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and } r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

These values of the two roots show that the **character** of these roots, as **real** or **imaginary, rational** or **irrational, equal** or **unequal,** depends upon the value of $\sqrt{b^2 - 4ac}$.

1. *To determine whether the roots are real or imaginary.*

If the quantity under the radical sign is positive, the roots are real, if negative, the roots are imaginary.

Hence, *when $b^2 - 4ac$ is positive, the roots are real,*
when $b^2 - 4ac$ is negative, the roots are imaginary.

2. *To determine whether the roots are rational or irrational.*

If the roots are imaginary, they are also irrational. If the roots are real, they may be rational or irrational, according to whether the quantity under the radical sign is a perfect square or not.

Hence, *when $b^2 - 4ac$ is a perfect square or equal to zero, the roots are rational; otherwise, they are irrational.*

3. *To determine whether the roots are equal or unequal.*

It is evident that the two roots r_1 and r_2 differ from each other in that the radical is added to $-b$ in the one and subtracted from $-b$ in the other. If the radical vanishes, the difference between the two roots vanishes and the roots are equal.

Hence, *when* $b_2 - 4ac$ *equals zero, the roots are equal.*

The expression $b^2 - 4ac$ is called the **discriminant** of the quadratic equation $ax^2 + bx + c = 0$.

4. *To determine the signs of the roots.*

When a is positive, b and c either positive or negative, the signs of the roots r_1 and r_2 may be determined from the signs of b and c.

If *c is positive,* $\sqrt{b^2 - 4ac} < b$, hence *both roots have the sign of* $-b$.

If *c is negative,* $\sqrt{b^2 - 4ac} > b$, and *the roots will have opposite signs,* since r_1 will be positive and r_2 negative.

If *b is negative,* $-b$ is positive and therefore r_1 *will be numerically greater than* r_2.

If *b is positive,* $-b$ is negative and therefore r_2 *will be numerically greater than* r_1.

In general, *if c is negative, the roots have opposite signs and the numerically greater root has the sign opposite to that of b.*

Determine the character of the roots of $2x^2 + 3x = 5$.

Reduce the equation to the general form,

$$2x^2 + 3x - 5 = 0$$

$$\text{hence } \left.\begin{array}{l} a = 2 \\ b = 3 \\ c = -5 \end{array}\right\} b^2 - 4ac = 9 + 40 = 49$$

As $b^2 - 4ac$ is a perfect square, the roots are real and rational.

As $b^2 - 4ac$ is not equal to zero, the roots are unequal.

As c is negative, the roots have opposite signs, and as b is positive, the negative root is the numerically greater.

EXERCISE 135

Without solving, determine the character of the roots of:

1. $3x^2 - 2x = 8$.
2. $5x^2 - 6x = 27$.
3. $2x^2 - 5x = 7$.
4. $3x^2 + 7x = 6$.
5. $5x^2 - 7x = 24$.
6. $8x^2 + 3x = 26$.
7. $7x^2 + 5x = 150$.
8. $6x^2 + 5x = 14$.
9. $8x^2 + 7x = 51$.
10. $7x^2 - 20x = 75$.
11. $11x^2 - 10x = 24$.
12. $4x^2 - 23x + 30 = 0$.

13. For what values of m wil $x^2 + 2mx = -4$ have equal roots?

In the equation, $x^2 + 2mx + 4 = 0$,

When $b^2 - 4ac = 0$, the roots are equal.

$$\left.\begin{array}{l} a = 1 \\ b = 2m \\ c = 4 \end{array}\right\} \begin{array}{r} \text{Therefore } 4m^2 - 16 = 0 \\ 4m^2 = +16 \\ m^2 = 4 \\ m = \pm 2 \end{array}$$

14. For what values of a will the equation $2x^2 + 3ax + 2 = 0$, have equal roots?

15. For what values of m will the equation $4x^2 + (6 - 2a)x = -1$ have equal roots?

16. For what values of m will the equation $(m + 1)x^2 + (m - 1)x + m + 1 = 0$ have imaginary roots?

RELATION OF ROOTS AND COEFFICIENTS

185. Any quadratic equation in the form $ax^2 + bx + c = 0$, may be reduced to the form, $x^2 + px + q = 0$, by dividing by the coefficient of x^2.

By solving $x^2 + px + q = 0$, we find

$$r_1 = \frac{-p + \sqrt{p^2 - 4q}}{2} \text{ and } r_2 = \frac{-p - \sqrt{p^2 - 4q}}{2}$$

by addition, $r_1 + r_2 = -\frac{2p}{2} = -p$

by multiplication, $r_1r_2 = \frac{p^2 - (p^2 - 4q)}{4} = q$.

Hence, *the sum of the roots of a quadratic equation in the form* $x^2 + px + q = 0$, *is equal to the coefficient of* x *with its sign changed, and the product of the roots is equal to the absolute term.*

As the above principle corresponds exactly to the rule for writing the product of two binomials, the equation $x^2 + px + q = 0$, may be written $(x - r_1)(x - r_2) = 0$.

Therefore, when the roots are known, *the equation may be* *formed by subtracting each root from x and placing the product of the remainders equal to zero.*

EXERCISE 136

1. Form an equation whose roots are $(-1 \pm \sqrt{-2})$.

Subtract each root from x, and write the product equal to zero.

$$\begin{aligned}(x + 1 - \sqrt{-2})(x + 1 + \sqrt{-2}) &= 0\\ (x + 1)^2 - (\sqrt{-2})^2 &= 0\\ x^2 + 2x + 1 + 2 &= 0\\ x^2 + 2x + 3 &= 0\end{aligned}$$

Form the equation whose roots are:

2. $3, 5$.
3. $\frac{1}{2}, \frac{1}{3}$.
4. $6, -8$.
5. $-2, -\frac{2}{3}$.
6. $-\frac{1}{2}, -\frac{3}{4}$.
7. $a, -3a$.
8. $2 - a, 2 + a$.
9. a, b.
10. $\sqrt{a} + b, \sqrt{a} - b$.
11. $\frac{1}{2}(2 \pm \sqrt{3})$.
12. $2 + \sqrt{-3}. 2 - \sqrt{-3}$.
13. $a(1 \pm \sqrt{3})$.
14. $(m + n), (m - n)$.
15. $\frac{1}{3}(3 \pm 5\sqrt{2})$.

GRAPHS OF HIGHER EQUATIONS

186. The graphs of equations or functions of a higher degree than the second, may be plotted by the methods already shown. This graphical method furnishes a convenient method of locating the real roots of higher equations.

In general, the number of real roots of an equation in x will be equal to the number of times the graph cuts the X axis. When the graph touches the X axis without crossing it, that is, when the graph is tangent to the X axis, equal roots are indicated. If the graph approaches the X axis and then recedes without touching the axis, the roots are imaginary.

FR.^s VIETE.
né en 1540. mort en 1603.

FRANCISCUS VIETA

Born at Fontenay-le-Comte, 1540. Died at Paris, 1603.

The greatest French algebraist of the sixteenth century.

He was a lawyer, who took up mathematics as a relaxation.

His *In Artem Analyticam Isagoge*, 1591, is the first work with a symbolic treatment of algebra.

Vieta arrived at an incomplete understanding of the relations between the coefficients and roots of an equation.

He also wrote on geometry and trigonometry.

EXERCISE 137

1. Locate the roots of $x^3 - 3x^2 + 4x - 2 = 0$.

Let $y = x^3 - 3x^2 + 4x - 2$, then any value of x that makes $y = 0$ will be a root of the equation; hence any point of the graph that lies on the X axis indicates a root of the equation.

x	y
0	-2
1	0
2	2
3	10
4	30
5	68
-1	-10

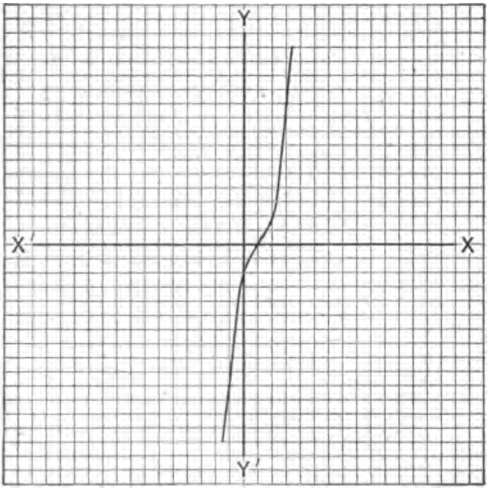


FIG. I.

For any negative value of x, y is negative, hence the curve cannot touch the X axis on the left of the Y axis, and as for positive values of x greater than 2, the value of y becomes positively greater, the graph goes away from the X axis. Hence, there is only one real root, $x = 1$, the other two roots of this cubic equation are imaginary. (Fig. I.)

2. Locate the roots of $x^3 - 8x + 3 = 0$.

Let $y = x^3 - 8x + 3$ (Fig. II).

x	y
0	3
1	-4
2	-5
3	6
4	35
-1	10
-2	11
-3	0
-4	-29

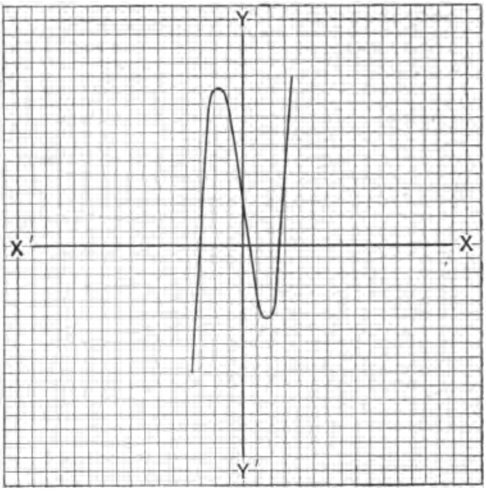


Fig. II.

Hence the roots are at -3, between 0 and 1, and between 2 and 3.

Locate, graphically, the real roots of:

3. $x^3 - 10x - 3 = 0$.
4. $x^3 - 26x + 5 = 0$.
5. $x^3 - 15x - 4 = 0$.
6. $x^3 + 2x^2 + 9 = 0$.
7. $x^3 - 3x + 2 = 0$.
8. $x^3 + 9x^2 + 16x + 4 = 0$.
9. $x^3 - 2x^2 - 23x + 60 = 0$.
10. $x^3 + 5x^2 + 7x + 2 = 0$.

REVIEW EXERCISE XXVI

Form the equation whose roots are:

1. $3 + \sqrt{6}$ and $3 - \sqrt{6}$.
2. $\dfrac{\sqrt{a} + 1}{2}$ and $\dfrac{\sqrt{a} - 1}{2}$.
3. $2 + \sqrt{-3}$ and $2 - \sqrt{-3}$.
4. $2 + 5\sqrt{3}$ and $2 - 5\sqrt{3}$.
5. $1 + \sqrt{-1}$ and $1 - \sqrt{-1}$.

6. $\frac{8}{11}$ and $\frac{2}{11}$.
7. 2 and $-\frac{8}{5}$.
8. $\frac{m+\sqrt{n}}{2}$ and $\frac{m-\sqrt{n}}{2}$.
9. $-2+\sqrt{3}$ and $-2-\sqrt{3}$.
10. $2b$ and $\frac{c}{b}$.

Without solving, discuss the roots of the equation:

11. $3x^2 + 17x - 5 = 0$.
12. $10x^2 - 21x - 10 = 0$.
13. $25x^2 = 10x - 1$.
14. $3x^2 - 5x + 4 = 0$.
15. $\frac{7}{x} - \frac{15}{x+2} + \frac{5}{x-8} = 0$.
16. $4x^2 - 7x = 10$.
17. $2x^2 - x + 5 = 0$.
18. $2x - x = 1$.
19. $7x^2 + 13x = 5$.
20. $x^2 - 3x + 5 = 0$.

21. Determine the values of k that the equation $(k+2)x^2 - 10kx + 25 = 0$ may have equal roots.

22. Form the quadratic equation, one of whose roots is $3+\sqrt{2}$, and the sum of whose roots is 6.

23. For what values of m are the roots of $2mx^2 + 7mx = x^2 + 5x - 5m$ equal?

24. Form the quadratic equation in which one root is $3+\sqrt{2}$ and the product of the roots is 7.

25. Locate the roots of the equation, $x^3 - 3x^2 - 9x + 1 = 0$.

26. Form an equation whose roots shall be the cubes of the roots of the equation $2x(x-a) = a^2$.

27. Under what conditions will the equation $x^2 + px = q$ have roots that are reciprocals of each other?

28. Locate the roots of $x^3 + 4x^2 - 7 = 0$.

CHAPTER XXVII

PERMUTATIONS AND COMBINATIONS

187. All the different **orders** in which it is possible to arrange a given number of things, by taking either some or all at a time, are called the **permutations** of the things.

Thus, the permutations of the figures 2 and 3, are 23, 32; the permutations of three figures, 2, 3, 4, taking two at a time, are 23, 24, 32, 42, 34, 43.

All the different **selections** that it is possible to make from a given number of things, taking either some or all at a time, *without regard to the order* in which they are arranged, are called the **combinations** of the things.

Thus, the combinations of three letters *a*, *b*, *c*, taken two at a time are *ab*, *ac*, *bc*, where the order of the letters is disregarded. Each of these combinations will furnish two permutations, as, *ab*, *ba*; *ac*, *ca*; *bc*, *cb*.

When all the things are taken, there can be only one combination, as *abc* of the three letters *a*, *b*, *c*.

Notation.—The symbol for the number of permutations of n things taking r at a time, is P^n_r, of n different things taking n at a time, is P^n_n, of 6 different things taking 3 at a time is P^6_3.

Sometimes, instead of P^n_r, ${}^nP_r$ or $P_{n,r}$ is used.

The symbol for the number of combinations of n different things, taking r at a time, is C^n_r; of n different things, taking n at a time, is C^n_n.

These are also sometimes written ${}^nC_r$ or $C_{n,r}$ and ${}^nC_n$ or $C_{n,n}$.

188. To find the number of permutations of n different things taken *r* at a time.

The permutations of *a*, *b*, *c*, taken two at a time, are

ab ba ca
ac bc cb

evidently formed by writing after each of the three letters each of the other letters. Therefore the total number of permutations is 3×2.

The number of permutations of n letters, taken 2 at a time, is found in the same way, by writing after each of the n letters, each of the remaining $n - 1$ letters. Therefore the total number of permutations of n letters, taken 2 at a time, is $n\ (n - 1)$.

If the n letters are taken 3 at a time, the permutations may be obtained by adding each of the remaining $n - 2$ letters to each permutation of the letters taken 2 at a time, or the total number of permutations of n letters, taken 3 at a time, is $n(n - 1)(n - 2)$. Hence,

The number of permutations of n different things, taken r at a time, is equal to the continued product of he natural numbers from n down to n − (r − 1) inclusive, the number of factors being equal to r.

$$\therefore P_r^n = n(n - 1)(n - 2).....(n - r + 1) \quad (1)$$

To get the factors from n down to 1, we must supply the factors $(n - r)(n - r - 1)(n - r - 2).....2.1.$

If the right member of equation (1) is multiplied and divided by this product, which is $\underline{|n - r}$ (factorial $n - r$ means the factors from 1 to $n - r$), we obtain,

$$P_r^n = \frac{\underline{|n}}{\underline{|n - r}} \quad (2).$$

Equation (1) is usually more convenient in the solution of numerical problems.

When all the n things are taken at a time, $r = n$, and equation (1) becomes,

$$P_n^n = \underline{|n} \quad (3).$$

EXERCISE 138

1. In how many orders may 5 persons sit on a bench?

Substitute in the formula, $P_5^5 = \underline{|5}$

$= 5 \times 4 \times 3 \times 2 \times 1 = 120$

2. Three persons enter a public telephone station in which there are 6 empty booths. In how many ways may they select booths?

This is evidently a problem to find the number of permutations that may be made of 6 things taken 3 at a time.

Hence $P_3^6 = 6 \times 5 \times 4 = 120$.

3. How many numbers of 4 figures each can be formed with the figures 1, 2, 3, 4?

4. How many permutations may be made of the letters in the word *grapes*?

5. Four persons board a boat on which there are 10 vacant seats. In how many ways may they choose seats?

6. In how many different ways may 10 books be arranged on a shelf?

7. If 8 dogs compete at a dog show, in how many ways may the blue and red ribbons be awarded?

189. To find the number of combinations of n different things taken r at a time.

If from the number of permutations of n things taken r at a time, we subtract the number of permutations made from combinations by changing the order of arrangement, we should have left the actual number of combinations of the n things, taken r at a time.

Thus, since two letters, as a and b, have two permutations, ab, ba, but only one combination, the number of combinations of n things, taken 2 at a time, will be only half the number of permutations of n things, taken 2 at a time.

Similarly, since three letters, taken three at a time, have 3×2 permutations, but only 1 combination, the number of combinations of n things, taken 3 at a time, is found by dividing the number of permutations of n things, taken 3 at a time, by 3×2.

In general then, since the number of permutations of n things taken n at a time is $\underline{|n}$, but the number of combinations only 1,

The number of combinations of n different things, taken

r at a time, is equal to the number of permutations of n things, taken r at a time, divided by the number of permutations of r things taken r at a time.

Hence $C_r^n = P_r^n \div P_r^r$

$$= \frac{n(n-1)(n-2)\dots(n-r+1)}{1\cdot2\cdot3\dots r} \qquad (4)$$

or $$C_r^n = \frac{\underline{|n}}{\underline{|n-r}} \div \underline{|r}$$

$$= \frac{\underline{|n}}{\underline{|r}\ \underline{|n-r}} \qquad (5)$$

Since for every combination of r things taken from n different things, there is left a combination of $n-r$ things,

The number of combinations of n different things, taken r at a time, is equal to the number of combinations when taken $n-r$ at a time.

Hence $$C_r^n = C_{n-r}^n = \frac{\underline{|n}}{\underline{|r}\ \underline{|n-r}} \qquad (6)$$

EXERCISE 139

1. In how many ways may a committee of three be selected from a club of 10 members?

As a committee is not changed by arranging the members of the committee in different orders, this is a problem in combinations.

Substituting in the formula, $C_r^n = \dfrac{\underline{|n}}{\underline{|r}\ \underline{|n-r}}$

$$C_3^{10} = \frac{\underline{|10}}{\underline{|3}\ \underline{|7}}$$

$\underline{|7}$ cancels all the factors from 7 to 1,

Hence $C_3^{10} = \dfrac{10\cdot\overset{3}{\cancel{9}}\cdot\overset{4}{\cancel{8}}}{\cancel{2}\cdot\cancel{3}} = 120$

2. In how many ways may a committee of 8, consisting of 5 Republicans and 3 Democrats, be selected from 10 Republicans and 7 Democrats?

It is evident that 5 Republicans may be selected from 10 Republicans in C_5^{10} different ways.

Also 3 Democrats may be selected from 7 Democrats in C_3^7 different ways.

But as any selection of Republicans may be combined with any selection of Democrats to form this committee of 8, the total number of committees is found by multiplying C_5^{10} by C_3^7.

Hence $$C_5^{10} \times C_3^7 = \frac{\underline{|10}}{\underline{|5}\;\underline{|5}} \times \frac{\underline{|7}}{\underline{|3}\;\underline{|4}}$$
$$= \frac{10\cdot9\cdot8\cdot7\cdot6}{2\cdot3\cdot4\cdot5} \times \frac{7\cdot6\cdot5}{2\cdot3} = 8820$$

3. From 7 consonants and 4 vowels, how many words may be formed, each word consisting of 4 consonants and 2 vowels, if any arrangement of the letters is considered a word?

The number of combinations of 4 consonants is C_4^7 and the number of combinations of 2 vowels is C_2^4.

Each combination of consonants may be combined with each combination of vowels, hence $C_4^7 \times C_2^4$.

But each combination of 6 letters, 4 consonants and 2 vowels, has $\underline{|6}$ permutations.

Hence, the total number of words is $C_4^7 \times C_2^4 \times \underline{|6}$.

4. If $6C_3^n = 4C_4^{n+1}$, find n.

$$6C_3^n = \frac{6\,\underline{|n}}{\underline{|3}\;\underline{|n-3}} \qquad 4C_4^{n+1} = \frac{4\,\underline{|n+1}}{\underline{|4}\;\underline{|n-3}}$$

$$\therefore \frac{6\,\underline{|n}}{\underline{|3}\;\underline{|n-3}} = \frac{4\,\underline{|n+1}}{\underline{|4}\;\underline{|n-3}}$$

Divide by $\underline{|n}$, $$\frac{6}{\underline{|3}\;\underline{|n-3}} = \frac{4(n+1)}{\underline{|4}\;\underline{|n-3}}$$

Multiply by $\underline{|n-3}$, $$\frac{6}{\underline{|3}} = \frac{4(n+1)}{\underline{|4}}$$

Clear of fractions, $6\,\underline{|4} = 4(n+1)\,\underline{|3}$

Divide by $\underline{|3}$, $6\cdot4 = 4(n+1)$

$n + 1 = 6 \quad \therefore n = 5$

5. In how many ways may a man select a dinner party of 5 from 8 friends?

6. In how many ways may a football eleven be selected from 20 candidates?

7. How many different hands of 13 cards each may be dealt from a deck of 52 cards?

8. From 8 Republicans and 6 Democrats, how many different committees may be selected, each composed of 3 Republicans and 2 Democrats?

9. From 15 consonants and 5 vowels how many words containing 3 consonants and 2 vowels may be formed, if any arrangement of the letters is considered a word?

10. Fifty points are situated in a plane, no three of the points in a straight line. How many triangles may be formed by connecting the points with straight lines?

11. In a house, 12 windows are available for watching a parade, 6 on the second floor and 6 on the third. If 5 of the windows are occupied, 3 on the second floor and 2 on the third, in how many ways may 7 more persons be accommodated, one at each window?

12. If $C_2^n = \frac{1}{2}C_5^n$ find the value of n.

13. How many different sums may be paid with a cent, a 5-cent piece, a dime, a quarter and a half dollar?

14. How many words may be formed from the word *Edgar*, if each word begins with E and any arrangement of the letters is considered a word?

190. To find the number of permutations of n things, taken n at a time, when they are not all different.

Suppose that there are n letters, a, b, c, d, etc., of which a occurs r times, b occurs p times, c occurs q times, and the other letters occur only once.

Let N stand for the required number of permutations. It is evident, that in any arrangement of letters, the interchange of like letters will not change the form of the arrangement. Hence, if the r letters a, were all different instead of alike, since r different letters have $\underline{|r}$ permutations, the actual number of permutations of the n letters would be $N \times \underline{|r}$.

Similarly, if the p letters b were all different, since p letters have $\underline{|p}$ permutations, the actual number of permutations of the n letters would be $N \times \underline{|r} \times \underline{|p}$. And if the q letters c were all different, the number of permutations of the n letters would be $N \times \underline{|r} \times \underline{|p} \times \underline{|q}$.

Since, in that case, the n letters are all different, $P_n^n = \underline{|n}$.

Therefore $N \times \underline{|r} \times \underline{|p} \times \underline{|q} = \underline{|n}$.

$$\text{or} \qquad N = \frac{\underline{|n}}{\underline{|r}\,\underline{|p}\,\underline{|q}}$$

That is, *The number of permutations of n things, taken all at a time, when r of them are of one kind, p of another, and q of another, and so on, is* $\dfrac{\underline{|n}}{\underline{|r}\,\underline{|p}\,\underline{|q}\cdots}$

EXERCISE 140

1. How many permutations may be made with the letters of the word *Connecticut*, taken all together?

$n = 11$, there are 3 c's, 2 n's, and 2 t's.

$$\therefore\ N = \frac{\underline{|11}}{\underline{|3}\,\underline{|2}\,\underline{|2}} = \frac{11 \times 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times \cancel{4}}{\cancel{2} \times \cancel{2}}$$
$$= 1{,}663{,}200$$

2. How many permutations may be made with the letters of the word *Roosevelt* taken all together?

3. How many different signals can be made with 3 yellow flags, 2 red flags, 5 green flags and 4 blue flags, used all together?

4. How many different numbers can be formed by using the six figures, 1, 1, 2, 2, 3, 3?

191. To find the total number of combinations of n different things.

The **total** number of combinations of n different things is the number of combinations of n different things taken successively, 1, 2, 3, ... n at a time.

Thus, the total number of combinations of 2 things is, $C_1^2 + C_2^2 = 2 + 1 = 3$, which may be expressed $2^2 - 1$.

The total number of combinations of 3 things is

$C_1^3 + C_2^3 + C_3^3 = 3 + 3 + 1 = 7$ or $2^3 - 1$.

The total number of combinations of 4 things is,

$$C_1^4 + C_2^4 + C_3^4 + C_4^4 = 4 + 6 + 4 + 1 = 15 \text{ or } 2^4 - 1,$$

and so on.

Hence: *The total number of combinations of n different things is equal to* $2^n - 1$.

EXERCISE 141

1. How many different sums may be made with a cent, a dime, a quarter, a half dollar and a dollar?

Since $C_{\text{total}}^n = 2^n - 1$

and $n = 5$, $C_{\text{total}}^5 = 2^5 - 1 = 31$

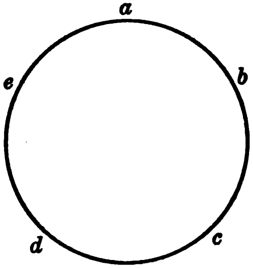
2. How many signals may be made with 7 different flags?

3. How many different quantities may be weighed with the weights, ½ oz., 1 oz., ½ lb. 1 lb. 2 lbs. 5 lbs.?

4. A man belongs to a club of ten members. In how many different ways may he invite one or more of the members to dinner?

192. To find the number of circular permutations of *n* different things taken all together.

If 5 letters a, b, c, d, e were placed in position around a circle in the order *abcde*, the arrangement could be read *abcde, bcdea, cdeab, deabc, eabcd.* Hence one circular arrangement corresponds to five straight line arrangements, that is, the number of circular permutations of 5 different things is one-fifth of the number of permutations of 5 different things taken all together. Similarly, *the number of*



circular permutations of n things will be $\frac{1}{n}$th the number of permutations of n different things taken all together.

Hence $P^n_{\text{circular}} = P_n^n \div n = \frac{\underline{|n}}{n} = \underline{|n-1}.$

EXERCISE 142

1. In how many ways may 5 persons seat themselves around a table?

2. In how many orders may 4 couples seat themselves around a table?

3. In how many ways may 4 gentlemen and their wives seat themselves around a table so that each gentleman sits opposite his wife?

4. In how many ways may 4 gentlemen and their wives seat themselves around a table so that each gentleman sits opposite a lady?

5. In how many ways may the colors, violet, indigo, blue, green, yellow, orange and red, be arranged on a disk, the colors radiating from the center?

REVIEW EXERCISE XXVII

1. In how many ways may 10 be thrown with two dice?

2. In how many different ways may 15 balls be arranged in a row, if 7 of the balls are white, 5 black and 3 red?

3. How many different sums can be paid with a cent, a nickel, a dime, and a quarter?

4. In how many different ways may six students be seated in ten seats?

5. How many baseball nines can be made up of 12 players, the pitcher and catcher of each nine being the same?

6. How many different committees, each consisting of 2 Republicans and 3 Democrats, can be formed from 14 Republicans and 21 Democrats?

7. If 8 points lie in a plane and no three of them in a straight line, how many straight lines can be drawn connecting two points?

8. In a party of 4 ladies and 3 gentlemen, a game of tennis is to be arranged, each side consisting of one gentleman and one lady. In how many ways may this be done?

9. How many different signals can be given with 4 red flags, 3 blue flags, and 3 white flags, each signal requiring all flags?

10. In how many ways can one mail 3 letters in a village containing 5 letter boxes?

11. How many numbers of four figures each can be formed from the figures, 1, 2, 3, 4, 5, 6, 7?

12. Supposing dice were constructed with 10 faces, how many different throws could be made with three dice?

13. A guard of 5 men must be selected every night out of a detachment of 32 men. For how many nights can a different guard be selected and how many times will each man serve?

14. In how many ways may 4 gentlemen and 4 ladies sit at a round table, so that no two gentlemen sit together?

15. A bus can accomodate 5 passengers on each side. In how many ways can 10 persons take seats, when a certain two always sit on one side and a certain one always on the other?

16. In a certain school, 20 boys come out for a practice game of football. In how many ways may the coach select a trial eleven? After the eleven is selected, he assigns 4 boys to play the four positions back of the line, but the 7 boys in the line he decides to interchange among the seven positions. In how many ways may the team be arranged?

17. A man has 5 coats, 6 vests, and 8 pairs of trousers. In how many different suits can he appear?

18. In how many ways may 7 books be arranged on a shelf, so that two particular books will not be together?

19. How may words can be formed from the letters of the word *education*, provided the second, fourth, sixth, and last letters are always consonants?

20. In how many different ways can 4 bridge hands of 13 cards each be dealt from a pack of 52 cards?

A FIVE PLACE TABLE
OF THE
COMMON LOGARITHMS
OF
NUMBERS

No.	0	1	2	3	4	5	6	7	8	9	D
100	00 000	00 043	00 087	00 130	00 173	00 217	00 260	00 303	00 346	00 389	43
101	00 432	00 475	00 518	00 561	00 604	00 647	00 689	00 732	00 775	00 817	43
102	00 860	00 903	00 945	00 988	01 030	01 072	01 115	01 157	01 199	01 242	42
103	01 284	01 326	01 368	01 410	01 452	01 494	01 536	01 578	01 620	01 662	42
104	01 703	01 745	01 787	01 828	01 870	01 912	01 953	01 995	02 036	02 078	42
105	02 119	02 160	02 202	02 243	02 284	02 325	02 366	02 407	02 449	02 490	41
106	02 531	02 572	02 612	02 653	02 694	02 735	02 776	02 816	02 857	02 898	41
107	02 938	02 979	03 019	03 060	03 100	03 141	03 181	03 222	03 262	03 302	40
108	03 342	03 383	03 423	03 463	03 503	03 543	03 583	03 623	03 663	03 703	40
109	03 743	03 782	03 822	03 862	03 902	03 941	03 981	04 021	04 060	04 100	40
110	04 139	04 179	04 218	04 258	04 297	04 336	04 376	04 415	04 454	04 493	39
111	04 532	04 571	04 610	04 650	04 689	04 727	04 766	04 805	04 844	04 883	39
112	04 922	04 961	04 999	05 038	05 077	05 115	05 154	05 192	05 231	05 269	39
113	05 308	05 346	05 385	05 423	05 461	05 500	05 538	05 576	05 614	05 652	38
114	05 690	05 729	05 767	05 805	05 843	05 881	05 918	05 956	05 994	06 032	38
115	06 070	06 108	06 145	06 183	06 221	06 258	06 296	06 333	06 371	06 408	38
116	06 446	06 483	06 521	06 558	06 595	06 633	06 670	06 707	06 744	06 781	37
117	06 819	08 856	06 893	06 930	06 967	07 004	07 041	07 078	07 115	07 151	37
118	07 188	07 225	07 262	07 298	07 335	07 372	07 408	07 445	07 482	07 518	37
119	07 555	07 591	07 628	07 664	07 700	07 737	07 773	07 809	07 846	07 882	36
120	07 918	07 954	07 990	08 027	08 063	08 099	08 135	08 171	08 207	08 243	36
121	08 279	08 314	08 350	08 386	08 422	08 458	08 493	08 529	08 565	08 600	36
122	08 636	08 672	08 707	08 743	08 778	08 814	08 849	08 884	08 920	08 955	35
123	08 991	09 026	09 061	09 096	09 132	09 167	09 202	09 237	09 272	09 307	35
124	09 342	09 377	09 412	09 447	09 482	09 517	09 552	09 587	09 621	09 656	35
125	09 691	09 726	09 760	09 795	09 830	09 864	09 899	09 934	09 968	10 003	35
126	10 037	10 072	10 106	10 140	10 175	10 209	10 243	10 278	10 312	10 346	34
127	10 380	10 415	10 449	10 483	10 517	10 551	10 585	10 619	10 653	10 687	34
128	10 721	10 755	10 789	10 823	10 857	10 890	10 924	10 958	10 992	11 025	34
129	11 059	11 093	11 126	11 160	11 193	11 227	11 261	11 294	11 327	11 361	34
130	11 394	11 428	11 461	11 494	11 528	11 561	11 594	11 628	11 661	11 694	33
131	11 727	11 760	11 793	11 826	11 860	11 893	11 926	11 959	11 992	12 024	33
132	12 057	12 090	12 123	12 156	12 189	12 222	12 254	12 287	12 320	12 352	33
133	12 385	12 418	12 450	12 483	12 516	12 548	12 581	12 613	12 646	12 678	33
134	12 710	12 743	12 775	12 808	12 840	12 872	12 905	12 937	12 969	13 001	32
135	13 033	13 066	13 098	13 130	13 162	13 194	13 226	13 258	13 290	13 322	32
136	13 354	13 386	13 418	13 450	13 481	13 513	13 545	13 577	13 609	13 640	32
137	13 672	13 704	13 735	13 767	13 799	13 830	13 862	13 893	13 925	13 956	32
138	13 988	14 019	14 051	14 082	14 114	14 145	14 176	14 208	14 239	14 270	31
139	14 301	14 333	14 364	14 395	14 426	14 457	14 489	14 520	14 551	14 582	31
140	14 613	14 644	14 675	14 706	14 737	14 768	14 799	14 829	14 860	14 891	31
141	14 922	14 953	14 983	15 014	15 045	15 076	15 106	15 137	15 168	15 198	31
142	15 229	15 259	15 290	15 320	15 351	15 381	15 412	15 442	15 473	15 503	31
143	15 534	15 564	15 594	15 625	15 655	15 685	15 715	15 746	15 776	15 806	30
144	15 836	15 866	15 897	15 927	15 957	15 987	16 017	16 047	16 077	16 107	30
145	16 137	16 167	16 197	16 227	16 256	16 286	16 316	16 346	16 376	16 406	30
146	16 435	16 465	16 495	16 524	16 554	16 584	16 613	16 643	16 673	16 702	30
147	16 732	16 761	16 791	16 820	16 850	16 879	16 909	16 938	16 967	16 997	29
148	17 026	17 056	17 085	17 114	17 143	17 173	17 202	17 231	17 260	17 289	29
149	17 319	17 348	17 377	17 406	17 435	17 464	17 493	17 522	17 551	17 580	29

No.	0	1	2	3	4	5	6	7	8	9	D
150	17 609	17 638	17 667	17 696	17 725	17 754	17 782	17 811	17 840	17 869	29
151	17 898	17 926	17 955	17 984	18 013	18 041	18 070	18 099	18 127	18 156	29
152	18 184	18 213	18 241	18 270	18 298	18 327	18 355	18 384	18 412	18 441	29
153	18 469	18 498	18 526	18 554	18 583	18 611	18 639	18 667	18 696	18 724	28
154	18 752	18 780	18 808	18 837	18 865	18 893	18 921	18 949	18 977	19 005	28
155	19 033	19 061	19 089	19 117	19 145	19 173	19 201	19 229	19 257	19 285	28
156	19 312	19 340	19 368	19 396	19 424	19 451	19 479	19 507	19 535	19 562	28
157	19 590	19 618	19 645	19 673	19 700	19 728	19 756	19 783	19 811	19 838	28
158	19 866	19 893	19 921	19 948	19 976	20 003	20 030	20 058	20 085	20 112	27
159	20 140	20 167	20 194	20 222	20 249	20 276	20 303	20 330	20 358	20 385	27
160	20 412	20 439	20 466	20 493	20 520	20 548	20 575	20 602	20 629	20 656	27
161	20 683	20 710	20 737	20 763	20 790	20 817	20 844	20 871	20 898	20 925	27
162	20 952	20 978	21 005	21 032	21 059	21 085	21 112	21 139	21 165	21 192	27
163	21 219	21 245	21 272	21 299	21 325	21 352	21 378	21 405	21 431	21 458	27
164	21 484	21 511	21 537	21 564	21 590	21 617	21 643	21 669	21 696	21 722	26
165	21 748	21 775	21 801	21 827	21 854	21 880	21 906	21 932	21 958	21 985	26
166	22 011	22 037	22 063	22 089	22 115	22 141	22 167	22 194	22 220	22 246	26
167	22 272	22 298	22 324	22 350	22 376	22 401	22 427	22 453	22 479	22 505	26
168	22 531	22 557	22 583	22 608	22 634	22 660	22 686	22 712	22 737	22 763	26
169	22 789	22 814	22 840	22 866	22 891	22 917	22 943	22 968	22 994	23 019	26
170	23 045	23 070	23 096	23 121	23 147	23 172	23 198	23 223	23 249	23 274	25
171	23 300	23 325	23 350	23 376	23 401	23 426	23 452	23 477	23 502	23 528	25
172	23 553	23 578	23 603	23 629	23 654	23 679	23 704	23 729	23 754	23 779	25
173	23 805	23 830	23 855	23 880	23 905	23 930	23 955	23 980	24 005	24 030	25
174	24 055	24 080	24 105	24 130	24 155	24 180	24 204	24 229	24 254	24 279	25
175	24 304	24 329	24 353	24 378	24 403	24 428	24 452	24 477	24 502	24 527	25
176	24 551	24 576	24 601	24 625	24 650	24 674	24 699	24 724	24 748	24 773	25
177	24 797	24 822	24 846	24 871	24 895	24 920	24 944	24 969	24 993	25 018	25
178	25 042	25 066	25 091	25 115	25 139	25 164	25 188	25 212	25 237	25 261	24
179	25 285	25 310	25 334	25 358	25 382	25 406	25 431	25 455	25 479	25 503	24
180	25 527	25 551	25 575	25 600	25 624	25 648	25 672	25 696	25 720	25 744	24
181	25 768	25 792	25 816	25 840	25 864	25 888	25 912	25 935	25 959	25 983	24
182	26 007	26 031	26 055	26 079	26 102	26 126	26 150	26 174	26 198	26 221	24
183	26 245	26 269	26 293	26 316	26 340	26 364	26 387	26 411	26 435	26 458	24
184	26 482	26 505	26 529	26 553	26 576	26 600	26 623	26 647	26 670	26 694	24
185	26 717	26 741	26 764	26 788	26 811	26 834	26 858	26 881	26 905	26 928	23
186	26 951	26 975	26 998	27 021	27 045	27 068	27 091	27 114	27 138	27 161	23
187	27 184	27 207	27 231	27 254	27 277	27 300	27 323	27 346	27 370	27 393	23
188	27 416	27 439	27 462	27 485	27 508	27 531	27 554	27 577	27 600	27 623	23
189	27 646	27 669	27 692	27 715	27 738	27 761	27 784	27 807	27 830	27 852	23
190	27 875	27 898	27 921	27 944	27 967	27 989	28 012	28 035	28 058	28 081	23
191	28 103	28 126	28 149	28 171	28 194	28 217	28 240	28 262	28 285	28 307	23
192	28 330	28 353	28 375	28 398	28 421	28 443	28 466	28 488	28 511	28 533	23
193	28 556	28 578	28 601	28 623	28 646	28 668	28 691	28 713	28 735	28 758	22
194	28 780	28 803	28 825	28 847	28 870	28 892	28 914	28 937	28 959	28 981	22
195	29 003	29 026	29 048	29 070	29 092	29 115	29 137	29 159	29 181	29 203	22
196	29 226	29 248	29 270	29 292	29 314	29 336	29 358	29 380	29 403	29 425	22
197	29 447	29 469	29 491	29 513	29 535	29 557	29 579	29 601	29 623	29 645	22
198	29 667	29 688	29 710	29 732	29 754	29 776	29 798	29 820	29 842	29 863	22
199	29 885	29 907	29 929	29 951	29 973	29 994	30 016	30 038	30 060	30 081	22

No.	0	1	2	3	4	5	6	7	8	9	D
200	30 103	30 125	30 146	30 168	30 190	30 211	30 233	30 255	30 276	30 298	22
201	30 320	30 341	30 363	30 384	30 406	30 428	30 449	30 471	30 492	30 514	22
202	30 535	30 557	30 578	30 600	30 621	30 643	30 664	30 685	30 707	30 728	21
203	30 750	30 771	30 792	30 814	30 835	30 856	30 878	30 899	30 920	30 942	21
204	30 963	30 984	31 006	31 027	31 048	31 069	31 091	31 112	31 133	31 154	21
205	31 175	31 197	31 218	31 239	31 260	31 281	31 302	31 323	31 345	31 366	21
206	31 387	31 408	31 429	31 450	31 471	31 492	31 513	31 534	31 555	31 576	21
207	31 597	31 618	31 639	31 660	31 681	31 702	31 723	31 744	31 765	31 785	21
208	31 806	31 827	31 848	31 869	31 890	31 911	31 931	31 952	31 973	31 994	21
209	32 015	32 035	32 056	32 077	32 098	32 118	32 139	32 160	32 181	32 201	21
210	32 222	32 243	32 263	32 284	32 305	32 325	32 346	32 366	32 387	32 408	21
211	32 428	32 449	32 469	32 490	32 510	32 531	32 552	32 572	32 593	32 613	20
212	32 634	32 654	32 675	32 695	32 715	32 736	32 756	32 777	32 797	32 818	20
213	32 838	32 858	32 879	32 899	32 919	32 940	32 960	32 980	33 001	33 021	20
214	33 041	33 062	33 082	33 102	33 122	33 143	33 163	33 183	33 203	33 224	20
215	33 244	33 264	33 284	33 304	33 325	33 345	33 365	33 385	33 405	33 425	20
216	33 445	33 465	33 486	33 506	33 526	33 546	33 566	33 586	33 606	33 626	20
217	33 646	33 666	33 686	33 706	33 726	33 746	33 766	33 786	33 806	33 826	20
218	33 846	33 866	33 885	33 905	33 925	33 945	33 965	33 985	34 005	34 025	20
219	34 044	34 064	34 084	34 104	34 124	34 143	34 163	34 183	34 203	34 223	20
220	34 242	34 262	34 282	34 301	34 321	34 341	34 361	34 380	34 400	34 420	20
221	34 439	34 459	34 479	34 498	34 518	34 537	34 557	34 577	34 596	34 616	20
222	34 635	34 655	34 674	34 694	34 713	34 733	34 753	34 772	34 792	34 811	19
223	34 830	34 850	34 869	34 889	34 908	34 928	34 947	34 967	34 986	35 005	19
224	35 025	35 044	35 064	35 083	35 102	35 122	35 141	35 160	35 180	35 199	19
225	35 218	35 238	35 257	35 276	35 295	35 315	35 334	35 353	35 372	35 392	19
226	35 411	35 430	35 449	35 468	35 488	35 507	35 526	35 545	35 564	35 583	19
227	35 603	35 622	35 641	35 660	35 679	35 698	35 717	35 736	35 755	35 774	19
228	35 793	35 813	35 832	35 851	35 870	35 889	35 908	35 927	35 946	35 965	19
229	35 984	36 003	36 021	36 040	36 059	36 078	36 097	30 116	36 135	36 154	19
230	36 173	36 192	36 211	36 229	36 248	36 267	36 286	36 305	36 324	36 342	19
231	36 361	36 380	36 399	36 418	36 436	36 455	36 474	36 493	36 511	36 530	19
232	36 549	36 568	36 586	36 605	36 624	36 642	36 661	36 680	36 698	36 717	19
233	36 736	36 754	36 773	36 791	36 810	36 829	36 847	36 866	36 884	36 903	19
234	36 922	36 940	36 959	36 977	36 996	37 014	37 033	37 051	37 070	37 088	18
235	37 107	37 125	37 144	37 162	37 181	37 199	37 218	37 236	37 254	37 273	18
236	37 291	37 310	37 328	37 346	37 365	37 383	37 401	37 420	37 438	37 457	18
237	37 475	37 493	37 511	37 530	37 548	37 566	37 585	37 603	37 621	37 639	18
238	37 658	37 676	37 694	37 712	37 731	37 749	37 767	37 785	37 803	37 822	18
239	37 840	37 858	37 876	37 894	37 912	37 931	37 949	37 967	37 985	38 003	18
240	38 021	38 039	38 057	38 075	38 093	38 112	38 130	38 148	38 166	38 184	18
241	38 202	38 220	38 238	38 256	38 274	38 292	38 310	38 328	38 346	38 364	18
242	38 382	38 399	38 417	38 435	38 453	38 471	38 489	38 507	38 525	38 543	18
243	38 561	38 578	38 596	38 614	38 632	38 650	38 668	38 686	38 703	38 721	18
244	38 739	38 757	38 775	38 792	38 810	38 828	38 846	38 863	38 881	38 899	18
245	38 917	38 934	38 952	38 970	38 987	39 005	39 023	39 041	39 058	39 076	18
246	39 094	39 111	39 129	39 146	39 164	39 182	39 199	39 217	39 235	39 252	18
247	39 270	39 287	39 305	39 322	39 340	39 358	39 375	39 393	39 410	39 428	18
248	39 445	39 463	39 480	39 498	39 515	39 533	39 550	39 568	39 585	39 602	17
249	39 620	39 637	39 655	39 672	39 690	39 707	39 724	39 742	39 759	39 777	17

No.	0	1	2	3	4	5	6	7	8	9	D
250	39 794	39 811	39 829	39 846	39 863	39 881	39 898	39 915	39 933	39 950	17
251	39 967	39 985	40 002	40 019	40 037	40 054	40 071	40 088	40 106	40 123	17
252	40 140	40 157	40 175	40 192	40 209	40 226	40 243	40 261	40 278	40 295	17
253	40 312	40 329	40 346	40 364	40 381	40 398	40 415	40 432	40 449	40 466	17
254	40 483	40 500	40 518	40 535	40 552	40 569	40 586	40 603	40 620	40 637	17
255	40 654	40 671	40 688	40 705	40 722	40 739	40 756	40 773	40 790	40 807	17
256	40 824	40 841	40 858	40 875	40 892	40 909	40 926	40 943	40 960	40 976	17
257	40 993	41 010	41 027	41 044	41 061	41 078	41 095	41 111	41 128	41 145	17
258	41 162	41 179	41 196	41 212	41 229	41 246	41 263	41 280	41 296	41 313	17
259	41 330	41 347	41 363	41 380	41 397	41 414	41 430	41 447	41 464	41 481	17
260	41 497	41 514	41 531	41 547	41 564	41 581	41 597	41 614	41 631	41 647	17
261	41 664	41 681	41 697	41 714	41 731	41 747	41 764	41 780	41 797	41 814	17
262	41 830	41 847	41 863	41 880	41 896	41 913	41 929	41 946	41 963	41 979	16
263	41 996	42 012	42 029	42 045	42 062	42 078	42 095	42 111	42 127	42 144	16
264	42 160	42 177	42 193	42 210	42 226	42 243	42 259	42 275	42 292	42 308	16
265	42 325	42 341	42 357	42 374	42 390	42 406	42 423	42 439	42 455	42 472	16
266	42 488	42 504	42 521	42 537	42 553	42 570	42 586	42 602	42 619	42 635	16
267	42 651	42 667	42 684	42 700	42 716	42 732	42 749	42 765	42 781	42 797	16
268	42 813	42 830	42 846	42 862	42 878	42 894	42 911	42 927	42 943	42 959	16
269	42 975	42 991	43 008	43 024	43 040	43 056	43 072	43 088	43 104	43 120	16
270	43 136	43 152	43 169	43 185	43 201	43 217	43 233	43 249	43 265	43 281	16
271	43 297	43 313	43 329	43 345	43 361	43 377	43 393	43 409	43 425	43 441	16
272	43 457	43 473	43 489	43 505	43 521	43 537	43 553	43 569	43 584	43 600	16
273	43 616	43 632	43 648	43 664	43 680	43 696	43 712	43 727	43 743	43 759	16
274	43 775	43 791	43 807	43 823	43 838	43 854	43 870	43 886	43 902	43 917	16
275	43 933	43 949	43 965	43 981	43 996	44 012	44 028	44 044	44 059	44 075	16
276	44 091	44 107	44 122	44 138	44 154	44 170	44 185	44 201	44 217	44 232	16
277	44 248	44 264	44 279	44 295	44 311	44 326	44 342	44 358	44 373	44 389	16
278	44 404	44 420	44 436	44 451	44 467	44 483	44 498	44 514	44 529	44 545	16
279	44 560	44 576	44 592	44 607	44 623	44 638	44 654	44 669	44 685	44 700	16
280	44 716	44 731	44 747	44 762	44 778	44 793	44 809	44 824	44 840	44 855	15
281	44 871	44 886	44 902	44 917	44 932	44 948	44 963	44 979	44 994	45 010	15
282	45 025	45 040	45 056	45 071	45 086	45 102	45 117	45 133	45 148	45 163	15
283	45 179	45 194	45 209	45 225	45 240	45 255	45 271	45 286	45 301	45 317	15
284	45 332	45 347	45 362	45 378	45 393	45 408	45 423	45 439	45 454	45 469	15
285	45 484	45 500	45 515	45 530	45 545	45 561	45 576	45 591	45 606	45 621	15
286	45 637	45 652	45 667	45 682	45 697	45 712	45 728	45 743	45 758	45 773	15
287	45 788	45 803	45 818	45 834	45 849	45 864	45 879	45 894	45 909	45 924	15
288	45 939	45 954	45 969	45 984	46 000	46 015	46 030	46 045	46 060	46 075	15
289	46 090	46 105	46 120	46 135	46 150	46 165	46 180	46 195	46 210	46 225	15
290	46 240	46 255	46 270	46 285	46 300	46 315	46 330	46 345	46 359	46 374	15
291	46 389	46 404	46 419	46 434	46 449	46 464	46 479	46 494	46 509	46 523	15
292	46 538	46 553	46 568	46 583	46 598	46 613	46 627	46 642	46 657	46 672	15
293	46 687	46 702	46 716	46 731	46 746	46 761	46 776	46 790	46 805	46 820	15
294	46 835	46 850	46 864	46 879	46 894	46 909	46 923	46 938	46 953	46 967	15
295	46 982	46 997	47 012	47 026	47 041	47 056	47 070	47 085	47 100	47 114	15
296	47 129	47 144	47 159	47 173	47 188	47 202	47 217	47 232	47 246	47 261	15
297	47 276	47 290	47 305	47 319	47 334	47 349	47 363	47 378	47 392	47 407	15
298	47 422	47 436	47 451	47 465	47 480	47 494	47 509	47 524	47 538	47 553	15
299	47 567	47 582	47 596	47 611	47 625	47 640	47 654	47 669	47 683	47 698	15

No.	0	1	2	3	4	5	6	7	8	9	D
300	47 712	47 727	47 741	47 756	47 770	47 784	47 799	47 813	47 828	47 842	14
301	47 857	47 871	47 885	47 900	47 914	47 929	47 943	47 958	47 972	47 986	14
302	48 001	48 015	48 029	48 044	48 058	48 073	48 087	48 101	48 116	48 130	14
303	48 144	48 159	48 173	48 187	48 202	48 216	48 230	48 244	48 259	48 273	14
304	48 287	48 302	48 316	48 330	48 344	48 359	48 373	48 387	48 401	48 416	14
305	48 430	48 444	48 458	48 473	48 487	48 501	48 515	48 530	45 544	48 558	14
306	48 572	48 586	48 601	48 615	48 629	48 643	48 657	48 671	48 686	48 700	14
307	48 714	48 728	48 742	48 756	48 770	48 785	48 799	48 813	48 827	48 841	14
308	48 855	48 869	48 883	48 897	48 911	48 926	48 940	48 954	48 968	48 982	14
309	48 996	49 010	49 024	49 038	49 052	49 066	49 080	49 094	49 108	49 122	14
310	49 136	49 150	49 164	49 178	49 192	49 206	49 220	49 234	49 248	49 262	14
311	49 276	49 290	49 304	49 318	49 332	49 346	49 360	49 374	49 388	49 402	14
312	49 415	49 429	49 443	49 457	49 471	49 485	49 499	49 513	49 527	49 541	14
313	49 554	49 568	49 582	49 596	49 610	49 624	49 638	49 651	49 665	49 679	14
314	49 693	49 707	49 721	49 734	49 748	49 762	49 776	49 790	49 803	49 817	14
315	49 831	49 845	49 859	49 872	49 886	49 900	49 914	49 927	49 941	49 955	14
316	49 969	49 982	49 996	50 010	50 024	50 037	50 051	50 065	50 079	50 092	14
317	50 106	50 120	50 133	50 147	50 161	50 174	50 188	50 202	50 215	50 229	14
318	50 243	50 256	50 270	50 284	50 297	50 311	50 325	50 338	50 352	50 365	14
319	50 379	50 393	50 406	50 420	50 433	50 447	50 461	50 474	50 488	50 501	14
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321	50 651	50 664	50 678	50 691	50 705	50 718	50 732	50 745	50 759	50 772	14
322	50 786	50 799	50 813	50 826	50 840	50 853	50 866	50 880	50 893	50 907	14
323	50 920	50 934	50 947	50 961	50 974	50 987	51 001	51 014	51 028	51 041	13
324	51 055	51 068	51 081	51 095	51 108	51 121	51 135	51 148	51 162	51 175	13
325	51 188	51 202	51 215	51 228	51 242	51 255	51 268	51 282	51 295	51 308	13
326	51 322	51 335	51 348	51 362	51 375	51 388	51 402	51 415	51 428	51 441	13
327	51 455	51 468	51 481	51 495	51 508	51 521	51 534	51 548	51 561	51 574	13
328	51 587	51 601	51 614	51 627	51 640	51 654	51 667	51 680	51 693	51 706	13
329	51 720	51 733	51 746	51 759	51 772	51 786	51 799	51 812	51 825	51 838	13
330	51 851	51 865	51 878	51 891	51 904	51 917	51 930	51 943	51 957	51 970	13
331	51 983	51 996	52 009	52 022	52 035	52 048	52 061	52 075	52 088	52 101	13
332	52 114	52 127	52 140	52 153	52 166	52 179	52 102	52 205	52 218	52 231	13
333	52 244	52 257	52 270	52 284	52 297	52 310	52 323	52 336	52 349	52 362	13
334	52 375	52 388	52 401	52 414	52 427	52 440	52 453	52 466	52 479	52 492	13
335	52 504	52 517	52 530	52 543	52 556	52 569	52 582	52 595	52 608	52 621	13
336	52 634	52 647	52 660	52 673	52 686	52 699	52 711	52 724	52 737	52 750	13
337	52 763	52 776	52 789	52 802	52 815	52 827	52 840	52 853	52 866	52 879	13
338	52 892	52 905	52 917	52 930	52 943	52 956	52 969	52 982	52 994	53 007	13
339	53 020	53 033	53 046	53 058	53 071	53 084	53 097	53 110	53 122	53 135	13
340	53 148	53 161	53 173	53 186	53 199	53 212	53 224	53 237	53 250	53 263	13
341	53 275	53 288	53 301	53 314	53 326	53 339	53 352	53 364	53 377	53 390	13
342	53 403	53 415	53 428	53 441	53 453	53 466	53 479	53 491	53 504	53 517	13
343	53 529	53 542	53 555	53 567	53 580	53 593	53 605	53 618	53 631	53 643	13
344	53 656	53 668	53 681	53 694	53 706	53 719	53 732	53 744	53 757	53 769	13
345	53 782	53 794	53 807	53 820	53 832	53 845	53 857	53 870	53 882	53 895	13
346	53 908	53 920	53 933	53 945	53 958	53 970	53 983	53 995	54 008	54 020	13
347	54 033	54 045	54 058	54 070	54 083	54 095	54 108	54 120	54 133	54 145	12
348	54 158	54 170	54 183	54 195	54 208	54 220	54 233	54 245	54 258	54 270	12
349	54 283	54 295	54 307	54 320	54 332	54 345	54 357	54 370	54 382	54 394	12

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350	54 407	54 419	54 432	54 444	54 456	54 469	54 481	54 494	54 506	54 518	12
351	54 531	54 543	54 555	54 568	54 580	54 593	54 605	54 617	54 630	54 642	12
352	54 654	54 667	54 679	54 691	54 704	54 716	54 728	54 741	54 753	54 765	12
353	54 777	54 790	54 802	54 814	54 827	54 839	54 851	54 864	54 876	54 888	12
354	54 900	54 913	54 925	54 937	54 949	54 962	54 974	54 986	54 998	55 011	12
355	55 023	55 035	55 047	55 060	55 072	55 084	55 096	55 108	55 121	55 133	12
356	55 145	55 157	55 169	55 182	55 194	55 206	55 218	55 230	55 242	55 255	12
357	55 267	55 279	55 291	55 303	55 315	55 328	55 340	55 352	55 364	55 376	12
358	55 388	55 400	55 413	55 425	55 437	55 449	55 461	55 473	55 485	55 497	12
359	55 509	55 522	55 534	55 546	55 558	55 570	55 582	55 594	55 606	55 618	12
360	55 630	55 642	55 654	55 666	55 678	55 691	55 703	55 715	55 727	55 739	12
361	55 751	55 763	55 775	55 787	55 799	55 811	55 823	55 835	55 847	55 859	12
362	55 871	55 883	55 895	55 907	55 919	55 931	55 943	55 955	55 967	55 979	12
363	55 991	56 003	56 015	56 027	56 038	56 050	56 062	56 074	56 086	56 098	12
364	56 110	56 122	56 134	56 146	56 158	56 170	56 182	56 194	56 205	56 217	12
365	56 229	56 241	56 253	56 265	56 277	56 289	56 301	56 312	56 324	56 336	12
366	56 348	56 360	56 372	56 384	56 396	56 407	56 419	56 431	56 443	56 455	12
367	56 467	56 478	56 490	56 502	56 514	56 526	56 538	56 549	56 561	56 573	12
368	56 585	56 597	56 608	56 620	56 632	56 644	56 656	56 667	56 679	56 691	12
369	56 703	56 714	56 726	56 738	56 750	56 761	56 773	56 785	56 797	56 808	12
370	56 820	56 832	56 844	56 855	56 867	56 879	56 891	56 902	56 914	56 926	12
371	56 937	56 949	56 961	56 972	56 984	56 996	57 008	57 019	57 031	57 043	12
372	57 054	57 066	57 078	57 089	57 101	57 113	57 124	57 136	57 148	57 159	12
373	57 171	57 183	57 194	57 206	57 217	57 229	57 241	57 252	57 264	57 276	12
374	57 287	57 299	57 310	57 322	57 334	57 345	57 357	57 368	57 380	57 392	12
375	57 403	57 415	57 426	57 438	57 449	57 461	57 473	57 484	57 496	57 507	12
376	57 519	57 530	57 542	57 553	57 565	57 576	57 588	57 600	57 611	57 623	12
377	57 634	57 646	57 657	57 669	57 680	57 692	57 703	57 715	57 726	57 738	12
378	57 749	57 761	57 772	57 784	57 795	57 807	57 818	57 830	57 841	57 852	11
379	57 864	57 875	57 887	57 898	57 910	57 921	57 933	57 944	57 955	57 967	11
380	57 978	57 990	58 001	58 013	58 024	58 035	58 047	58 058	58 070	58 081	11
381	58 092	58 104	58 115	58 127	58 138	58 149	58 161	58 172	58 184	58 195	11
382	58 206	58 218	58 229	58 240	58 252	58 263	58 274	58 286	58 297	58 309	11
383	58 320	58 331	58 343	58 354	58 365	58 377	58 388	58 399	58 410	58 422	11
384	58 433	58 444	58 456	58 467	58 478	58 490	58 501	58 512	58 524	58 535	11
385	58 546	58 557	58 569	58 580	58 591	58 602	58 614	58 625	58 636	58 647	11
386	58 659	58 670	58 681	58 692	58 704	58 715	58 726	58 737	58 749	58 760	11
387	58 771	58 782	58 794	58 805	58 816	58 827	58 838	58 850	58 861	58 872	11
388	58 883	58 894	58 906	58 917	58 928	58 939	58 950	58 961	58 973	58 984	11
389	58 995	59 006	59 017	59 028	59 040	59 051	59 062	59 073	59 084	59 095	11
390	59 106	59 118	59 129	59 140	59 151	59 162	59 173	59 184	59 195	59 207	11
391	59 218	59 229	59 240	59 251	59 262	59 273	59 284	59 295	59 306	59 318	11
392	59 329	59 340	59 351	59 362	59 373	59 384	59 395	59 406	59 417	59 428	11
393	59 439	59 450	59 461	59 472	59 483	59 494	59 506	59 517	59 528	59 539	11
391	59 550	59 561	59 572	59 583	59 594	59 605	59 616	59 627	59 638	59 649	11
395	59 660	59 671	59 682	59 693	59 704	59 715	59 726	59 737	59 748	59 759	11
396	59 770	59 780	59 791	59 802	59 813	59 824	59 835	59 846	59 857	59 868	11
397	59 879	59 890	59 901	59 912	59 923	59 934	59 945	59 956	59 966	59 977	11
398	59 988	59 999	60 010	60 021	60 032	60 043	60 054	60 065	60 076	60 086	11
399	60 097	60 108	60 119	60 130	60 141	60 152	60 163	60 173	60 184	60 195	11

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400	60 206	60 217	60 228	60 239	60 249	60 260	60 271	60 282	60 293	60 304	11
401	60 314	60 325	60 336	60 347	60 358	60 369	60 379	60 390	60 401	60 412	11
402	60 423	60 433	60 444	60 455	60 466	60 477	60 487	60 498	60 509	60 520	11
403	60 531	60 541	60 552	60 563	60 574	60 584	60 595	60 606	60 617	60 627	11
404	60 638	60 649	60 660	60 670	60 681	60 692	60 703	60 713	60 724	60 735	11
405	60 746	60 756	60 767	60 778	60 788	60 799	60 810	60 821	60 831	60 842	11
406	60 853	60 863	60 874	60 885	60 895	60 906	60 917	60 927	60 938	60 949	11
407	60 959	60 970	60 981	60 991	61 002	61 013	61 023	61 034	61 045	61 055	11
408	61 066	61 077	61 087	61 098	61 109	61 119	61 130	61 140	61 151	61 162	11
409	61 172	61 183	61 194	61 204	61 215	61 225	61 236	61 247	61 257	61 268	11
410	61 278	61 289	61 300	61 310	61 321	61 331	61 342	61 352	61 363	61 374	11
411	61 384	61 395	61 405	61 416	61 426	61 437	61 448	61 458	61 469	61 479	11
412	61 490	61 500	61 511	61 521	61 532	61 542	61 553	61 563	61 574	61 584	11
413	61 595	61 606	61 616	61 627	61 637	61 648	61 658	61 669	61 679	61 690	11
414	61 700	61 711	61 721	61 731	61 742	61 752	61 763	61 773	61 784	61 794	10
415	61 805	61 815	61 826	61 836	61 847	61 857	61 868	61 878	61 888	61 899	10
416	61 909	61 920	61 930	61 941	61 951	61 962	61 972	61 982	61 993	62 003	10
417	62 014	62 024	62 034	62 045	62 055	62 066	62 076	62 086	62 097	62 107	10
418	62 118	62 128	62 138	62 149	62 159	62 170	62 180	62 190	62 201	62 211	10
419	62 221	62 232	62 242	62 252	62 263	62 273	62 284	62 204	62 304	62 315	10
420	62 325	62 335	62 346	62 356	62 366	62 377	62 387	62 397	62 408	62 418	10
421	62 428	62 439	62 449	62 459	62 469	62 480	62 490	62 500	62 511	62 521	10
422	62 531	62 542	62 552	62 562	62 572	62 583	62 593	62 603	62 613	62 624	10
423	62 634	62 644	62 655	62 665	62 675	62 685	62 696	62 706	62 716	62 726	10
424	62 737	62 747	62 757	62 767	62 778	62 788	62 798	62 808	62 818	62 829	10
425	62 839	62 849	62 859	62 870	62 880	62 890	62 900	62 910	62 921	62 931	10
426	62 941	62 951	62 961	62 972	62 982	62 992	63 002	63 012	63 022	63 033	10
427	63 043	63 053	63 063	63 073	63 083	63 094	63 104	63 114	63 124	63 134	10
428	63 144	63 155	63 165	63 175	63 185	63 195	63 205	63 215	63 225	63 236	10
429	63 246	63 256	63 266	63 276	63 286	63 296	63 306	63 317	63 327	63 337	10
430	63 347	63 357	63 367	63 377	63 387	63 397	63 407	63 417	63 428	63 438	10
431	63 448	63 458	63 468	63 478	63 488	63 498	63 508	63 518	63 528	63 538	10
432	63 548	63 558	63 568	63 579	63 589	63 599	63 609	63 619	63 629	63 639	10
433	63 649	63 659	63 669	63 679	63 689	63 699	63 709	63 719	63 729	63 739	10
434	63 749	63 759	63 769	63 779	63 789	63 799	63 809	63 819	63 829	63 839	10
435	63 849	63 859	63 869	63 879	63 889	63 899	63 909	63 919	63 929	63 939	10
436	63 949	63 959	63 969	63 979	63 988	63 998	64 008	64 018	64 028	64 038	10
437	64 048	64 058	64 068	64 078	64 088	64 098	64 108	64 118	64 128	64 137	10
438	64 147	64 157	64 167	64 177	64 187	64 197	64 207	64 217	64 227	64 237	10
439	64 246	64 256	64 266	64 276	64 286	64 296	64 306	64 316	64 326	64 335	10
440	64 345	64 355	64 365	64 375	64 385	64 395	64 404	64 414	64 424	64 434	10
441	64 444	64 454	64 464	64 473	64 483	64 493	64 503	64 513	64 523	64 532	10
442	64 542	64 552	64 562	64 572	64 582	64 591	64 601	64 611	64 621	64 631	10
443	64 640	64 650	64 660	64 670	64 680	64 689	64 699	64 709	64 719	64 729	10
444	64 738	64 748	64 758	64 768	64 777	64 787	64 797	64 807	64 810	64 826	10
445	64 836	64 846	64 856	64 865	64 875	64 885	64 895	64 904	64 914	64 924	10
446	64 933	64 943	64 953	64 963	64 972	64 982	64 992	65 002	65 011	65 021	10
447	65 031	65 040	65 050	65 060	65 070	65 079	65 089	65 099	65 108	65 118	10
448	65 128	65 137	65 147	65 157	65 167	65 176	65 186	65 196	65 205	65 215	10
449	65 225	65 234	65 244	65 254	65 263	65 273	65 283	65 292	65 302	65 312	10

No.	0	1	2	3	4	5	6	7	8	9	D
450	65 321	65 331	65 341	65 350	65 360	65 369	65 379	65 389	65 398	65 408	10
451	65 418	65 427	65 437	65 447	65 456	65 466	65 475	65 485	65 495	65 504	10
452	65 514	65 523	65 533	65 543	65 552	65 562	65 571	65 581	65 591	65 600	10
453	65 610	65 619	65 629	65 639	65 648	65 658	65 667	65 677	65 686	65 696	10
454	65 706	65 715	65 725	65 734	65 744	65 753	65 763	65 772	65 782	65 792	10
455	65 801	65 811	65 820	65 830	65 839	65 849	65 858	65 868	65 877	65 887	9
456	65 896	65 906	65 916	65 925	65 935	65 944	65 954	65 963	65 973	65 982	9
457	65 992	66 001	66 011	66 020	66 030	66 039	66 049	66 058	66 068	66 077	9
458	66 087	66 096	66 106	66 115	66 124	66 134	66 143	66 153	66 162	66 172	9
459	66 181	66 191	66 200	66 210	66 219	66 229	66 238	66 247	66 257	66 266	9
460	66 276	66 285	66 295	66 304	66 314	66 323	66 332	66 342	66 351	66 361	9
461	66 370	66 380	66 389	66 398	66 408	66 417	66 427	66 436	66 445	66 455	9
462	66 464	66 474	66 483	66 492	66 502	66 511	66 521	66 530	66 539	66 549	9
463	66 558	66 567	66 577	66 586	66 596	66 605	66 614	66 624	66 633	66 642	9
464	66 652	66 661	66 671	66 680	66 689	66 699	66 708	66 717	66 727	66 736	9
465	66 745	66 755	66 764	66 773	66 783	66 792	66 801	66 811	66 820	66 829	9
466	66 839	66 848	66 857	66 867	66 876	66 885	66 894	66 904	66 913	66 922	9
467	66 932	66 941	66 950	66 960	66 969	66 978	66 987	66 997	67 006	67 015	9
468	67 025	67 034	67 043	67 052	67 062	67 071	67 080	67 089	67 099	67 108	9
469	67 117	67 127	67 136	67 145	67 154	67 164	67 173	67 182	67 191	67 201	9
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472	67 394	67 403	67 413	67 422	67 431	67 440	67 449	67 459	67 468	67 477	9
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474	67 578	67 587	67 596	67 605	67 614	67 624	67 633	67 642	67 651	67 660	9
475	67 669	67 679	67 688	67 697	67 706	67 715	67 724	67 733	67 742	67 752	9
476	67 761	67 770	67 779	67 788	67 797	67 806	67 815	67 825	67 834	67 843	9
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482	68 305	68 314	68 323	68 332	68 341	68 350	68 359	68 368	68 377	68 386	9
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484	68 485	68 494	68 502	68 511	68 520	68 529	68 538	68 547	68 556	68 565	9
485	68 574	68 583	68 592	68 601	68 610	68 619	68 628	68 637	68 646	68 655	9
486	68 664	68 673	68 681	68 690	68 699	68 708	68 717	68 726	68 735	68 744	9
487	68 753	68 762	68 771	68 780	68 789	68 797	68 806	68 815	68 824	68 833	9
488	68 842	68 851	68 860	68 869	68 878	68 886	68 895	68 904	68 913	68 922	9
489	68 931	68 940	68 949	68 958	68 966	68 975	68 984	68 993	69 002	69 011	9
490	69 020	69 028	69 037	69 046	69 055	69 064	69 073	69 082	69 090	69 099	9
491	69 108	69 117	69 126	69 135	69 144	69 152	69 161	69 170	69 179	69 188	9
492	69 197	69 205	69 214	69 223	69 232	69 241	69 249	69 258	69 267	69 276	9
493	69 285	69 294	69 302	69 311	69 320	69 329	69 338	69 346	69 355	69 364	9
494	69 373	69 381	69 390	69 399	69 408	69 417	69 425	69 434	69 443	69 452	9
495	69 461	69 469	69 478	69 487	69 496	69 504	69 513	69 522	69 531	69 539	9
496	69 548	69 557	69 566	69 574	69 583	69 592	69 601	69 609	69 618	69 627	9
497	69 636	69 644	69 653	69 662	69 671	69 679	69 688	69 697	69 705	69 714	9
498	69 723	69 732	69 740	69 749	69 758	69 767	69 775	69 784	69 793	69 801	9
499	69 810	69 819	69 827	69 836	69 845	69 854	69 862	69 871	69 880	69 888	9

No.	0	1	2	3	4	5	6	7	8	9	D
500	69 897	69 906	69 914	69 923	69 932	69 940	69 949	69 958	69 966	69 975	9
501	69 984	69 992	70 001	70 010	70 018	70 027	70 036	70 044	70 053	70 062	9
502	70 070	70 079	70 088	70 096	70 105	70 114	70 122	70 131	70 140	70 148	9
503	70 157	70 165	70 174	70 183	70 191	70 200	70 209	70 217	70 226	70 234	9
504	70 243	70 252	70 260	70 269	70 278	70 286	70 295	70 303	70 312	70 321	9
505	70 329	70 338	70 346	70 355	70 364	70 372	70 381	70 389	70 398	70 406	9
506	70 415	70 424	70 432	70 441	70 449	70 458	70 467	70 475	70 484	70 492	9
507	70 501	70 509	70 518	70 526	70 535	70 544	70 552	70 561	70 569	70 578	9
508	70 586	70 595	70 603	70 612	70 621	70 629	70 638	70 646	70 655	70 663	9
509	70 672	70 680	70 689	70 697	70 706	70 714	70 723	70 731	70 740	70 749	9
510	70 757	70 766	70 774	70 783	70 791	70 800	70 808	70 817	70 825	70 834	9
511	70 842	70 851	70 859	70 868	70 876	70 885	70 893	70 902	70 910	70 919	9
512	70 927	70 935	70 944	70 952	70 961	70 969	70 978	70 986	70 995	71 003	9
513	71 012	71 020	71 029	71 037	71 046	71 054	71 063	71 071	71 079	71 088	8
514	71 096	71 105	71 113	71 122	71 130	71 139	71 147	71 155	71 164	71 172	8
515	71 181	71 189	71 198	71 206	71 214	71 223	71 231	71 240	71 248	71 257	8
516	71 265	71 273	71 282	71 290	71 299	71 307	71 315	71 324	71 332	71 341	8
517	71 349	71 357	71 366	71 374	71 383	71 391	71 399	71 408	71 416	71 425	8
518	71 433	71 441	71 450	71 458	71 466	71 475	71 483	71 492	71 500	71 508	8
519	71 517	71 525	71 533	71 542	71 550	71 559	71 567	71 575	71 584	71 592	8
520	71 600	71 609	71 617	71 625	71 634	71 642	71 650	71 659	71 667	71 675	8
521	71 684	71 692	71 700	71 709	71 717	71 725	71 734	71 742	71 750	71 759	8
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523	71 850	71 858	71 867	71 875	71 883	71 892	71 900	71 908	71 917	71 925	8
524	71 933	71 941	71 950	71 958	71 966	71 975	71 983	71 991	71 999	72 008	8
525	72 016	72 024	72 032	72 041	72 049	72 057	72 066	72 074	72 082	72 090	8
526	72 099	72 107	72 115	72 123	72 132	72 140	72 148	72 156	72 165	72 173	8
527	72 181	72 189	72 198	72 206	72 214	72 222	72 230	72 239	72 247	72 255	8
528	72 263	72 272	72 280	72 288	72 296	72 304	72 313	72 321	72 329	72 337	8
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531	72 509	72 518	72 526	72 534	72 542	72 550	72 558	72 567	72 575	72 583	8
532	72 591	72 599	72 607	72 616	72 624	72 632	72 640	72 648	72 656	72 665	8
533	72 673	72 681	72 689	72 697	72 705	72 713	72 722	72 730	72 738	72 746	8
534	72 754	72 762	72 770	72 779	72 787	72 795	72 803	72 811	72 819	72 827	8
535	72 835	72 843	72 852	72 860	72 868	72 876	72 884	72 892	72 900	72 908	8
536	72 916	72 925	72 933	72 941	72 949	72 957	72 965	72 973	72 981	72 989	8
537	72 997	73 006	73 014	73 022	73 030	73 038	73 046	73 054	73 062	73 070	8
538	73 078	73 086	73 094	73 102	73 111	73 119	73 127	73 135	73 143	73 151	8
539	73 159	73 167	73 175	73 183	73 191	73 199	73 207	73 215	73 223	73 231	8
540	73 239	73 247	73 255	73 263	73 272	73 280	73 288	73 206	73 304	73 312	8
541	73 320	73 328	73 336	73 344	73 352	73 360	73 368	73 376	73 384	73 392	8
542	73 400	73 408	73 416	73 424	73 432	73 440	73 448	73 456	73 464	73 472	8
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544	73 560	73 568	73 576	73 584	73 592	73 600	73 608	73 616	73 624	73 632	8
545	73 640	73 648	73 656	73 664	73 672	73 679	73 687	73 695	73 703	73 711	8
546	73 719	73 727	73 735	73 743	73 751	73 759	73 767	73 775	73 783	73 791	8
547	73 799	73 807	73 815	73 823	73 830	73 838	73 846	73 854	73 862	73 870	8
548	73 878	73 886	73 894	73 902	73 910	73 918	73 926	73 933	73 941	73 949	8
549	73 957	73 965	73 073	73 981	73 989	73 997	74 005	73 013	74 020	74 028	8

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550	74 036	74 044	74 052	74 060	74 068	74 076	74 084	74 092	74 099	74 107	8
551	74 115	74 123	74 131	74 139	74 147	74 155	74 162	74 170	74 178	74 186	8
552	74 194	74 202	74 210	74 218	74 225	74 233	74 241	74 249	74 257	74 265	8
553	74 273	74 280	74 288	74 296	74 304	74 312	74 320	74 327	74 335	74 343	8
554	74 351	74 359	74 367	74 374	74 382	74 390	74 398	74 406	74 414	74 421	8
555	74 429	74 437	74 445	74 453	74 461	74 468	74 476	74 484	74 492	74 500	8
556	74 507	74 515	74 523	74 531	74 539	74 547	74 554	74 562	74 570	74 578	8
557	74 586	74 593	74 601	74 609	74 617	74 624	74 632	74 640	74 648	74 656	8
558	74 663	74 671	74 679	74 687	74 695	74 702	74 710	74 718	74 726	74 733	8
559	74 741	74 749	74 757	74 764	74 772	74 780	74 788	74 796	74 803	74 811	8
560	74 819	74 827	74 834	74 842	74 850	74 858	74 865	74 873	74 881	74 889	8
561	74 896	74 904	74 912	74 920	74 927	74 935	74 943	74 950	74 958	74 966	8
562	74 974	74 981	74 989	74 997	75 005	75 012	75 020	75 028	75 035	75 043	8
563	75 051	75 059	75 066	75 074	75 082	75 089	75 097	75 105	75 113	75 120	8
564	75 128	75 136	75 143	75 151	75 159	75 166	75 174	75 182	75 189	75 197	8
565	75 205	75 213	75 220	75 228	75 236	75 243	75 251	75 259	75 266	75 274	8
566	75 282	75 289	75 297	75 305	75 312	75 320	75 328	75 335	75 343	75 351	8
567	75 358	75 366	75 374	75 381	75 389	75 397	75 404	75 412	75 420	75 427	8
568	75 435	75 442	75 450	75 458	75 465	75 473	75 481	75 488	75 496	75 504	8
569	75 511	75 519	75 526	75 534	75 542	75 549	75 557	75 565	75 572	75 580	8
570	75 587	75 595	75 603	75 610	75 618	75 626	75 633	75 641	75 648	75 656	8
571	75 664	75 671	75 679	75 686	75 694	75 702	75 709	75 717	75 724	75 732	8
572	75 740	75 747	75 755	75 762	75 770	75 778	75 785	75 793	75 800	75 808	8
573	75 815	75 823	75 831	75 838	75 846	75 853	75 861	75 868	75 876	75 884	8
574	75 891	75 899	75 906	75 914	75 921	75 929	75 937	75 944	75 952	75 959	8
575	75 967	75 974	75 982	75 989	75 997	76 005	76 012	76 020	76 027	76 035	8
576	76 042	76 050	76 057	76 065	76 072	76 080	76 087	76 095	76 103	76 110	8
577	76 118	76 125	76 133	76 140	76 148	76 155	76 163	76 170	76 178	76 185	8
578	76 193	76 200	76 208	76 215	76 223	76 230	76 238	76 245	76 253	76 260	8
579	76 268	76 275	76 283	76 290	76 298	76 305	76 313	76 320	76 328	76 335	8
580	76 343	76 350	76 358	76 365	76 373	76 380	76 388	76 395	76 403	76 410	8
581	76 418	76 425	76 433	76 440	76 448	76 455	76 462	76 470	76 477	76 485	7
582	76 492	76 500	76 507	76 515	76 522	76 530	76 537	76 545	76 552	76 559	7
583	76 567	76 574	76 582	76 589	76 597	76 604	76 612	76 619	76 626	76 634	7
584	76 641	76 649	76 656	76 664	76 671	76 678	76 686	76 693	76 701	76 708	7
585	76 716	76 723	76 730	76 738	76 745	76 753	76 760	76 768	76 775	76 782	7
586	76 790	76 797	76 805	76 812	76 819	76 827	76 834	76 842	76 849	76 856	7
587	76 864	76 871	76 879	76 886	76 893	76 901	76 908	76 916	76 923	76 930	7
588	76 938	76 945	76 953	76 960	76 967	76 975	76 982	76 989	76 997	77 004	7
589	77 012	77 019	77 026	77 034	77 041	77 048	77 056	77 063	77 070	77 078	7
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591	77 159	77 166	77 173	77 181	77 188	77 195	77 203	77 210	77 217	77 225	7
592	77 232	77 240	77 247	77 254	77 262	77 269	77 276	77 283	77 291	77 298	7
593	77 305	77 313	77 320	77 327	77 335	77 342	77 349	77 357	77 364	77 371	7
594	77 379	77 386	77 393	77 401	77 408	77 415	77 422	77 430	77 437	77 444	7
595	77 452	77 459	77 466	77 474	77 481	77 488	77 495	77 503	77 510	77 517	7
596	77 525	77 532	77 539	77 546	77 554	77 561	77 568	77 576	77 583	77 590	7
597	77 597	77 605	77 612	77 619	77 627	77 634	77 641	77 648	77 656	77 663	7
598	77 670	77 677	77 685	77 692	77 699	77 706	77 714	77 721	77 728	77 735	7
599	77 743	77 750	77 757	77 764	77 772	77 779	77 786	77 793	77 801	77 808	7

No.	0	1	2	3	4	5	6	7	8	9	D
600	77 815	77 822	77 830	77 837	77 844	77 851	77 859	77 866	77 873	77 880	7
601	77 887	77 895	77 902	77 909	77 916	77 924	77 931	77 938	77 945	77 952	7
602	77 960	77 967	77 974	77 981	77 988	77 996	78 003	78 010	78 017	78 025	7
603	78 032	78 039	78 046	78 053	78 061	78 068	78 075	78 082	78 089	78 097	7
604	78 104	78 111	78 118	78 125	78 132	78 140	78 147	78 154	78 161	78 168	7
605	78 176	78 183	78 190	78 197	78 204	78 211	78 219	78 226	78 233	78 240	7
606	78 247	78 254	78 262	78 269	78 276	78 283	78 290	78 297	78 305	78 312	7
607	78 319	78 326	78 333	78 340	78 347	78 355	78 362	78 369	78 376	78 383	7
608	78 390	78 398	78 405	78 412	78 419	78 426	78 433	78 440	78 447	78 455	7
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611	78 604	78 611	78 618	78 625	78 633	78 640	78 647	78 654	78 661	78 668	7
612	78 675	78 682	78 689	78 696	78 704	78 711	78 718	78 725	78 732	78 739	7
613	78 746	78 753	78 760	78 767	78 774	78 781	78 789	78 796	78 803	78 810	7
614	78 817	78 824	78 831	78 838	78 845	78 852	78 859	78 866	78 873	78 880	7
615	78 888	78 895	78 902	78 909	78 916	78 923	78 930	78 937	78 944	78 951	7
616	78 958	78 965	78 972	78 979	78 986	78 993	79 000	79 007	79 014	79 021	7
617	79 029	79 036	79 043	79 050	79 057	79 064	79 071	79 078	79 085	79 092	7
618	79 099	79 106	79 113	79 120	79 127	79 134	79 141	79 148	79 155	79 162	7
619	79 169	79 176	79 183	79 190	79 197	79 204	79 211	79 218	79 225	79 232	7
620	79 239	79 246	79 253	79 260	79 267	79 274	79 281	79 288	79 295	79 302	7
621	79 309	79 316	79 323	79 330	79 337	79 344	79 351	79 358	79 365	79 372	7
622	79 379	79 386	79 393	79 400	79 407	79 414	79 421	79 428	79 435	79 442	7
623	79 449	79 456	79 463	79 470	79 477	79 484	79 491	79 498	79 505	79 511	7
624	79 518	79 525	79 532	79 539	79 546	79 553	79 560	79 567	79 574	79 581	7
625	79 588	79 595	79 602	79 609	79 616	79 623	79 630	79 637	79 644	79 650	7
626	79 657	79 664	79 671	79 678	79 685	79 692	79 699	79 706	79 713	79 720	7
627	79 727	79 734	79 741	79 748	79 754	79 761	79 768	79 775	79 782	79 789	7
628	79 796	79 803	79 810	79 817	79 824	79 831	79 837	79 844	79 851	79 858	7
629	79 865	79 872	79 879	79 886	79 893	79 900	79 906	79 913	79 920	79 927	7
630	79 934	79 941	79 948	79 955	79 962	79 969	79 975	79 982	79 989	79 996	7
631	80 003	80 010	80 017	80 024	80 030	80 037	80 044	80 051	80 058	80 065	7
632	80 072	80 079	80 085	80 092	80 099	80 106	80 113	80 120	80 127	80 134	7
633	80 140	80 147	80 154	80 161	80 168	80 175	80 182	80 188	80 195	80 202	7
634	80 209	80 216	80 223	80 229	80 236	80 243	80 250	80 257	80 264	80 271	7
635	80 277	80 284	80 291	80 298	80 305	80 312	80 318	80 325	80 332	80 339	7
636	80 346	80 353	80 359	80 366	80 373	80 380	80 387	80 393	80 400	80 407	7
637	80 414	80 421	80 428	80 434	80 441	80 448	80 455	80 462	80 468	80 475	7
638	80 482	80 489	80 496	80 502	80 509	80 516	80 523	80 530	80 536	80 543	7
639	80 550	80 557	80 564	80 570	80 577	80 584	80 591	80 598	80 604	80 611	7
640	80 618	80 625	80 632	80 638	80 645	80 652	80 659	80 665	80 672	80 679	7
641	80 686	80 693	80 699	80 706	80 713	80 720	80 726	80 733	80 740	80 747	7
642	80 754	80 760	80 767	80 774	80 781	80 787	80 794	80 801	80 808	80 814	7
643	80 821	80 828	80 835	80 841	80 848	80 855	80 862	80 868	80 875	80 882	7
644	80 889	80 895	80 902	80 909	80 916	80 922	80 929	80 936	80 943	80 949	7
645	80 956	80 963	80 969	80 976	80 983	80 990	80 996	81 003	81 010	81 017	7
646	81 023	81 030	81 037	81 043	81 050	81 057	81 064	81 070	81 077	81 084	7
647	81 090	81 097	81 104	81 111	81 117	81 124	81 131	81 137	81 144	81 151	7
648	81 158	81 164	81 171	81 178	81 184	81 191	81 198	81 204	81 211	81 218	7
649	81 224	81 231	81 238	81 245	81 251	81 258	81 265	81 271	81 278	81 285	7

No.	0	1	2	3	4	5	6	7	8	9	D
650	81 291	81 298	81 305	81 311	81 318	81 325	81 331	81 338	81 345	81 351	7
651	81 358	81 365	81 371	81 378	81 385	81 391	81 398	81 405	81 411	81 418	7
652	81 425	81 431	81 438	81 445	81 451	81 458	81 465	81 471	81 478	81 485	7
653	81 491	81 498	81 505	81 511	81 518	81 525	81 531	81 538	81 544	81 551	7
654	81 558	81 564	81 571	81 578	81 584	81 591	81 598	81 604	81 611	81 617	7
655	81 624	81 631	81 637	81 644	81 651	81 657	81 664	81 671	81 677	81 684	7
656	81 690	81 697	81 704	81 710	81 717	81 723	81 730	81 737	81 743	81 750	7
657	81 757	81 763	81 770	81 776	81 783	81 790	81 796	81 803	81 809	81 816	7
658	81 823	81 829	81 836	81 842	81 849	81 856	81 862	81 869	81 875	81 882	7
659	81 889	81 895	81 902	81 908	81 915	81 921	81 928	81 935	81 941	81 948	7
660	81 954	81 961	81 968	81 974	81 981	81 987	81 994	82 000	82 007	82 014	7
661	82 020	82 027	82 033	82 040	82 046	82 053	82 060	82 066	82 073	82 079	7
662	82 086	82 092	82 099	82 105	82 112	82 119	82 125	82 132	82 138	82 145	7
663	82 151	82 158	82 164	82 171	82 178	82 184	82 191	82 197	82 204	82 210	7
664	82 217	82 223	82 230	82 236	82 243	82 249	82 256	82 263	82 269	82 276	7
665	82 282	82 289	82 295	82 302	82 308	82 315	82 321	82 328	82 334	82 341	7
666	82 347	82 354	82 360	82 367	82 373	82 380	82 387	82 393	82 400	82 406	7
667	82 413	82 419	82 426	82 432	82 439	82 445	82 452	82 458	82 465	82 471	7
668	82 478	82 484	82 491	82 497	82 504	82 510	82 517	82 523	82 530	82 536	7
669	82 543	82 549	82 556	82 562	82 569	82 575	82 582	82 588	82 595	82 601	7
670	82 607	82 614	82 620	82 627	82 633	82 640	82 646	82 653	82 659	82 666	7
671	82 672	82 679	82 685	82 692	82 698	82 705	82 711	82 718	82 724	82 730	6
672	82 737	82 743	82 750	82 756	82 763	82 769	82 776	82 782	82 789	82 795	6
673	82 802	82 808	82 814	82 821	82 827	82 834	82 840	82 847	82 853	82 860	6
674	82 866	82 872	82 879	82 885	82 892	82 898	82 905	82 911	82 918	82 924	6
675	82 930	82 937	82 943	82 950	82 956	82 963	82 969	82 975	82 982	82 988	6
676	82 995	83 001	83 008	83 014	83 020	83 027	83 033	83 040	83 046	83 052	6
677	83 059	83 065	83 072	83 078	83 085	83 091	83 097	83 104	83 110	83 117	6
678	83 123	83 129	83 136	83 142	83 149	83 155	83 161	83 168	83 174	83 181	6
679	83 187	83 193	83 200	83 206	83 213	83 219	83 225	83 232	83 238	83 245	6
680	83 251	83 257	83 264	83 270	83 276	83 283	83 289	83 296	83 302	83 308	6
681	83 315	83 321	83 327	83 334	83 340	83 347	83 353	83 359	83 366	83 372	6
682	83 378	83 385	83 391	83 398	83 404	83 410	83 417	83 423	83 429	83 436	6
683	83 442	83 448	83 455	83 461	83 467	83 474	83 480	83 487	83 493	83 499	6
684	83 506	83 512	83 518	83 525	83 531	83 537	83 544	83 550	83 556	83 563	6
685	83 569	83 575	83 582	83 588	83 594	83 601	83 607	83 613	83 620	83 626	6
686	83 632	83 639	83 645	83 651	83 658	83 664	83 670	83 677	83 683	83 689	6
687	83 696	83 702	83 708	83 715	83 721	83 727	83 734	83 740	83 746	83 753	6
688	83 759	83 765	83 771	83 778	83 784	83 790	83 797	83 803	83 809	83 816	6
689	83 822	83 828	83 835	83 841	83 847	83 853	83 860	83 866	83 872	83 879	6
690	83 885	83 891	83 897	83 904	83 910	83 916	83 923	83 929	83 935	83 942	6
691	83 948	83 954	83 960	83 967	83 973	83 979	83 985	83 992	83 998	84 004	6
692	84 011	84 017	84 023	84 029	84 036	84 042	84 048	84 055	84 061	84 067	6
693	84 073	84 080	84 086	84 092	84 098	84 105	84 111	84 117	84 123	84 130	6
694	84 136	84 142	84 148	84 155	84 161	84 167	84 173	84 180	84 186	84 192	6
695	84 198	84 205	84 211	84 217	84 223	84 230	84 236	84 242	84 248	84 255	6
696	84 261	84 267	84 273	84 280	84 286	84 292	84 298	84 305	84 311	84 317	6
697	84 323	84 330	84 336	84 342	84 348	84 354	84 361	84 367	84 373	84 379	6
698	84 386	84 392	84 398	84 404	84 410	84 417	84 423	84 429	84 435	84 442	6
699	84 448	84 454	84 460	84 466	84 473	84 479	84 485	84 491	84 497	84 504	6

No.	0	1	2	3	4	5	6	7	8	9	D
700	84 510	84 516	84 522	84 528	84 535	84 541	84 547	84 553	84 559	84 566	6
701	84 572	84 578	84 584	84 590	84 597	84 603	84 609	84 615	84 621	84 628	6
702	84 634	84 640	84 646	84 652	84 658	84 665	84 671	84 677	84 683	84 689	6
703	84 696	84 702	84 708	84 714	84 720	84 726	84 733	84 739	84 745	84 751	6
704	84 757	84 763	84 770	84 776	84 782	84 788	84 794	84 800	84 807	84 813	6
705	84 819	84 825	84 831	84 837	84 844	84 850	84 856	84 862	84 868	84 874	6
706	84 880	84 887	84 893	84 899	84 905	84 911	84 917	84 924	84 930	84 936	6
707	84 942	84 948	84 954	84 960	84 967	84 973	84 979	84 985	84 991	84 997	6
708	85 003	85 009	85 016	85 022	85 028	85 034	85 040	85 046	85 052	85 058	6
709	85 065	85 071	85 077	85 083	85 089	85 095	85 101	85 107	85 114	85 120	6
710	85 126	85 132	85 138	85 144	85 150	85 156	85 163	85 169	85 175	85 181	6
711	85 187	85 193	85 199	85 205	85 211	85 217	85 224	85 230	85 236	85 242	6
712	85 248	85 254	85 260	85 266	85 272	85 278	85 285	85 291	85 297	85 303	6
713	85 309	85 315	85 321	85 327	85 333	85 339	85 345	85 352	85 358	85 364	6
714	85 370	85 376	85 382	85 388	85 394	85 400	85 406	85 412	85 418	85 425	6
715	85 431	85 437	85 443	85 449	85 455	85 461	85 467	85 473	85 479	85 485	6
716	85 491	85 497	85 503	85 509	85 516	85 522	85 528	85 534	85 540	85 546	6
717	85 552	85 558	85 564	85 570	85 576	85 582	85 588	85 594	85 600	85 606	6
718	85 612	85 618	85 625	85 631	85 637	85 643	85 649	85 655	85 661	85 667	6
719	85 673	85 679	85 685	85 691	85 697	85 703	85 709	85 715	85 721	85 727	6
720	85 733	85 739	85 745	85 751	85 757	85 763	85 769	85 775	85 781	85 788	6
721	85 794	85 800	85 806	85 812	85 818	85 824	85 830	85 836	85 842	85 848	6
722	85 854	85 860	85 866	85 872	85 878	85 884	85 890	85 896	85 902	85 908	6
723	85 914	85 920	85 926	85 932	85 938	85 944	85 950	85 956	85 962	85 968	6
724	85 974	85 980	85 986	85 992	85 998	86 004	86 010	86 016	86 022	86 028	6
725	86 034	86 040	86 046	86 052	86 058	86 064	86 070	86 076	86 082	86 088	6
726	86 094	86 100	86 106	86 112	86 118	86 124	86 130	86 136	86 141	86 147	6
727	86 153	86 159	86 165	86 171	86 177	86 183	86 189	86 195	86 201	86 207	6
728	86 213	86 219	86 225	86 231	86 237	86 243	86 249	86 255	86 261	86 267	6
729	86 273	86 279	86 285	86 291	86 297	86 303	86 308	86 314	86 320	86 326	6
730	86 332	86 338	86 344	86 350	86 356	86 362	86 368	86 374	86 380	86 386	6
731	86 392	86 398	86 404	86 410	86 415	86 421	86 427	86 433	86 439	86 445	6
732	86 451	86 457	86 463	86 469	86 475	86 481	86 487	86 493	86 499	86 504	6
733	86 510	86 516	86 522	86 528	86 534	86 540	86 546	86 552	86 558	86 564	6
734	86 570	86 576	86 581	86 587	86 593	86 599	86 605	86 611	86 617	86 623	6
735	86 629	86 635	86 641	86 646	86 652	86 658	86 664	86 670	86 676	86 682	6
736	86 688	86 694	86 700	86 705	86 711	86 717	86 723	86 729	86 735	86 741	6
737	86 747	86 753	86 759	86 764	86 770	86 776	86 782	86 788	86 794	86 800	6
738	86 806	86 812	86 817	86 823	86 829	86 835	86 841	86 847	86 853	85 859	6
739	86 864	86 870	86 876	86 882	86 888	86 894	86 900	86 906	86 911	86 917	6
740	86 923	86 929	86 935	86 941	86 947	86 953	86 958	86 964	86 970	86 976	6
741	86 982	86 988	86 994	86 999	87 005	87 011	87 017	87 023	87 029	87 035	6
742	87 040	87 046	87 052	87 058	87 064	87 070	87 075	87 081	87 087	87 093	6
743	87 099	87 105	87 111	87 116	87 122	87 128	87 134	87 140	87 146	87 151	6
744	87 157	87 163	87 169	87 175	87 181	87 186	87 192	87 198	87 204	87 210	6
745	87 216	87 221	87 227	87 233	87 239	87 245	87 251	87 256	87 262	87 268	6
746	87 274	87 280	87 286	87 291	87 297	87 303	87 309	87 315	87 320	87 326	6
747	87 332	87 338	87 344	87 349	87 355	87 361	87 367	87 373	87 379	87 384	6
748	87 390	87 396	87 402	87 408	87 413	87 419	87 425	87 431	87 437	87 442	6
749	87 448	87 454	87 460	87 466	87 471	87 477	87 483	87 489	87 495	87 500	6

No.	0	1	2	3	4	5	6	7	8	9	D
750	87 506	87 512	87 518	87 523	87 529	87 535	87 541	87 547	87 552	87 558	6
751	87 564	87 570	87 576	87 581	87 587	87 593	87 599	87 604	87 610	87 616	6
752	87 622	87 628	87 633	87 639	87 645	87 651	87 656	87 662	87 668	87 674	6
753	87 679	87 685	87 691	87 697	87 703	87 708	87 714	87 720	87 726	87 731	6
754	87 737	87 743	87 749	87 754	87 760	76 766	87 772	87 777	87 783	87 789	6
755	87 795	87 800	87 806	87 812	87 818	87 823	87 829	87 835	87 841	87 846	6
756	87 852	87 858	87 864	87 869	87 875	87 881	87 887	87 892	87 898	87 904	6
757	87 910	87 915	87 921	87 927	87 933	87 938	87 944	87 950	87 955	87 961	6
758	87 967	87 973	87 978	87 984	87 990	87 996	88 001	88 007	88 013	88 018	6
759	88 024	88 030	88 036	88 041	88 047	88 053	88 058	88 064	88 070	88 076	6
760	88 081	88 087	88 093	88 098	88 104	88 110	88 116	88 121	88 127	88 133	6
761	88 138	88 144	88 150	88 156	88 161	88 167	88 173	88 178	88 184	88 190	6
762	88 195	88 201	88 207	88 213	88 218	88 224	88 230	88 235	88 241	88 247	6
763	88 252	88 258	88 264	88 270	88 275	88 281	88 287	88 292	88 298	88 304	6
764	88 309	88 315	88 321	88 326	88 332	88 338	88 343	88 349	88 355	88 360	6
765	88 366	88 372	88 377	88 383	88 389	88 395	88 400	88 406	88 412	88 417	6
766	88 423	88 429	88 434	88 440	88 446	88 451	88 457	88 463	88 468	88 474	6
767	88 480	88 485	88 491	88 497	88 502	88 508	88 513	88 519	88 525	88 530	6
768	88 536	88 542	88 547	88 553	88 559	88 564	88 570	88 576	88 581	88 587	6
769	88 593	88 598	88 604	88 610	88 615	88 621	88 627	88 632	88 638	88 643	6
770	88 649	88 655	88 660	88 666	88 672	88 677	88 683	88 689	88 694	88 700	6
771	88 705	88 711	88 717	88 722	88 728	88 734	88 739	88 745	88 750	88 756	6
772	88 762	88 767	88 773	88 779	88 784	88 790	88 795	88 801	88 807	88 812	6
773	88 818	88 824	88 829	88 835	88 840	88 846	88 852	88 857	88 863	88 868	6
774	88 874	88 880	88 885	88 891	88 897	88 902	88 908	88 913	88 919	88 925	6
775	88 930	88 936	88 941	88 947	88 953	88 958	88 964	88 969	88 975	88 981	6
776	88 986	88 992	88 997	89 003	89 009	89 014	89 020	89 025	89 031	89 037	6
777	89 042	89 048	89 053	89 059	89 064	89 070	89 076	89 081	89 087	89 092	6
778	89 098	89 104	89 109	89 115	89 120	89 126	89 131	89 137	89 143	89 148	6
779	89 154	89 159	89 165	89 170	89 176	89 182	89 187	89 193	89 198	89 204	6
780	89 209	89 215	89 221	89 226	89 232	89 237	89 243	89 248	89 254	89 260	6
781	89 265	89 271	89 276	89 282	89 287	89 293	89 298	89 304	89 310	89 315	6
782	89 321	89 326	89 332	89 337	89 343	89 348	89 354	89 360	89 365	89 371	6
783	89 376	89 382	89 387	89 393	89 398	89 404	89 409	89 415	89 421	89 426	6
784	89 432	89 437	89 443	89 448	89 454	89 459	89 465	89 470	89 476	89 481	6
785	89 487	89 492	89 498	89 504	89 509	89 515	89 520	89 526	89 531	89 537	6
786	89 542	89 548	89 553	89 559	89 564	89 570	89 575	89 581	89 586	89 592	6
787	89 597	89 603	89 609	89 614	89 620	89 625	89 631	89 636	89 642	89 647	6
788	89 653	89 658	89 664	89 669	89 675	89 680	89 686	89 691	89 697	89 702	6
789	89 708	89 713	89 719	89 724	89 730	89 735	89 741	89 746	89 752	89 757	6
790	89 763	89 768	89 774	89 779	89 785	89 790	89 796	89 801	89 807	89 812	5
791	89 818	89 823	89 829	89 834	89 840	89 845	89 851	89 856	89 862	89 867	5
792	89 873	89 878	89 883	89 889	89 894	89 900	89 905	89 911	89 916	89 922	5
793	89 927	89 933	89 938	89 944	89 949	89 955	89 960	89 966	89 971	89 977	5
794	89 982	89 988	89 993	89 998	90 004	90 009	90 015	90 020	90 026	90 031	5
795	90 037	90 042	90 048	90 053	90 059	90 064	90 069	90 075	90 080	90 086	5
796	90 091	90 097	90 102	90 108	90 113	90 119	90 124	90 129	90 135	90 140	5
797	90 146	90 151	90 157	90 162	90 168	90 173	90 179	90 184	90 189	90 195	5
798	90 200	90 206	90 211	90 217	90 222	90 227	90 233	90 238	90 244	90 249	5
799	90 255	90 260	90 266	90 271	90 276	90 282	90 287	90 293	90 298	90 304	5

No.	0	1	2	3	4	5	6	7	8	9	D
800	90 309	90 314	90 320	90 325	90 331	90 336	90 342	90 347	90 352	90 358	5
801	90 363	90 369	90 374	90 380	90 385	90 390	90 396	90 401	90 407	90 412	5
802	90 417	90 423	90 428	90 434	90 439	90 445	90 450	90 455	90 461	90 466	5
803	90 472	90 477	90 482	90 488	90 493	90 499	90 504	90 509	90 515	90 520	5
804	90 526	90 531	90 536	90 542	90 547	90 553	90 558	90 563	90 569	90 574	5
805	90 580	90 585	90 590	90 596	90 601	90 607	90 612	90 617	90 623	90 628	5
806	90 634	90 639	90 644	90 650	90 655	90 660	90 666	90 671	90 677	90 682	5
807	90 687	90 693	90 698	90 703	90 709	90 714	90 720	90 725	90 730	90 736	5
808	90 741	90 747	90 752	90 757	90 763	90 768	90 773	90 779	90 784	90 789	5
809	90 795	90 800	90 806	90 811	90 816	90 822	90 827	90 832	90 838	90 843	5
810	90 849	90 854	90 859	90 865	90 870	90 875	90 881	90 886	90 891	90 897	5
811	90 902	90 907	90 913	90 918	90 924	90 929	90 934	90 940	90 945	90 950	5
812	90 956	90 961	90 966	90 972	90 977	90 982	90 988	90 993	90 998	91 004	5
813	91 009	91 014	91 020	91 025	91 030	91 036	91 041	91 046	91 052	91 057	5
814	91 062	91 068	91 073	91 078	91 084	91 089	91 094	91 100	91 105	91 110	5
815	91 116	91 121	91 126	91 132	91 137	91 142	91 148	91 153	91 158	91 164	5
816	91 169	91 174	91 180	91 185	91 190	91 196	91 201	91 206	91 212	91 217	5
817	91 222	91 228	91 233	91 238	91 243	91 249	91 254	91 259	91 265	91 270	5
818	91 275	91 281	91 286	91 291	91 297	91 302	91 307	91 312	91 318	91 323	5
819	91 328	91 334	91 339	91 344	91 350	91 355	91 360	91 365	91 371	91 376	5
820	91 381	91 387	91 392	91 397	91 403	91 408	91 413	91 418	91 424	91 429	5
821	91 434	91 440	91 445	91 450	91 455	91 461	91 466	91 471	91 477	91 482	5
822	91 487	91 492	91 498	91 503	91 508	91 514	91 519	91 524	91 529	91 535	5
823	91 540	91 545	91 551	91 556	91 561	91 566	91 572	91 577	91 582	91 587	5
824	91 593	91 598	91 603	91 609	91 614	91 619	91 624	91 630	91 635	91 640	5
825	91 645	91 651	91 656	91 661	91 666	91 672	91 677	91 682	91 687	91 693	5
826	91 698	91 703	91 709	91 714	91 719	91 724	91 730	91 735	91 740	91 745	5
827	91 751	91 756	91 761	91 766	91 772	91 777	91 782	91 787	91 793	91 798	5
828	91 803	91 808	91 814	91 819	91 824	91 829	91 834	91 840	91 845	91 850	5
829	91 855	91 861	91 866	91 871	91 876	91 882	91 887	91 892	91 897	91 903	5
830	91 908	91 913	91 918	91 924	91 929	91 934	91 939	91 944	91 950	91 955	5
831	91 960	91 965	91 971	91 976	91 981	91 986	91 991	91 997	92 002	92 007	5
832	92 012	92 018	92 023	92 028	92 033	92 038	92 044	92 049	92 054	92 059	5
833	92 065	92 070	92 075	92 080	92 085	92 091	92 096	92 101	92 106	92 111	5
834	92 117	92 122	92 127	92 132	92 137	92 143	92 148	92 153	92 158	92 163	5
835	92 169	92 174	92 179	92 184	92 189	92 195	92 200	92 205	92 210	92 215	5
836	92 221	92 226	92 231	92 236	92 241	92 247	92 252	92 257	92 262	92 267	5
837	92 273	92 278	92 283	92 288	92 293	92 298	92 304	92 309	92 314	92 319	5
838	92 324	92 330	92 335	92 340	92 345	92 350	92 355	92 361	92 366	92 371	5
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